Lecture Notes on Robustness and Opportuneness

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A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

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1 Preliminary Example: Reliability of a Beam With an Uncertain Load

(Source: Y. Ben-Haim, *Robust Reliability in the Mechanical Sciences,* sections 3.1, 3.2.)

¶ 3 components of reliability analysis:
1. A system model.
2. A failure criterion.
3. An uncertainty model.

¶ We will consider info-gap models of uncertainty and develop, in a preliminary example, the idea of **robust reliability.**

¶ Consider a:
- Uniform simply-supported beam.
- Uncertain distributed load density function, $\phi(x)$ [N/m].

¶ We wish to
- Analyze the reliability of the beam given very fragmentary information.
- Optimize the design of the beam by enhancing the reliability.
- Evaluate the impact of different levels and types of information.

¶ What we **do know** about the load:
- $\widetilde{\phi}(x)$ = nominal load density function, [N/m].
- Deviation from the nominal load is bounded along the beam.

¶ What we **do not know** about the load:
- The precise realization of the load density, $\phi(x)$.
- The bound on the deviation of the true from the nominal load.

¶ The disparity between what we
do know and what we **do not know**
is an **information gap.**

¶ Similarly, the disparity between what we
do know and what we **need to know**
for a fully competent design or analysis
is an **information gap.**

¶ We represent the load uncertainty with an info-gap model:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \le h \right\}, \quad h \ge 0 \tag{1}$$

This is an info-gap **uncertainty model.**

¶ Note the two levels of uncertainty in an info-gap model:
- At fixed h: true load profile $\phi(x)$ is unknown.
- Horizon of uncertainty — h — is unknown.

¶ **System model:**

- Static bending moment as a function of load profile: $M(x)$.
- For simple-simple beam one finds:

$$M(x) = -\frac{L-x}{L}\int_0^x \phi(u)u\,\mathrm{d}u - \frac{x}{L}\int_x^L \phi(u)(L-u)\,\mathrm{d}u \tag{2}$$

where L is the length of the beam.

¶ The **failure criterion:**

The beam fails if the bending moment $M(x)$
exceeds the critical value M_c:

$$\max_{0\le x\le L} |M(x)| > M_\mathrm{c} \tag{3}$$

¶ We evaluate the **robust reliability,** $\widehat{h}$, by combining
System model, uncertainty model, and failure criterion:
The robustness is:
The greatest info-gap, h,
such that the system model
does not violate the failure criterion
for any load profile up to uncertainty h.
We can express $\widehat{h}$ as:

$$\begin{aligned}
\widehat{h} &= \text{maximum tolerable uncertainty} && (4)\\
&= \max\{h: \ \text{failure cannot occur}\} && (5)\\
&= \max\left\{h: \ \left(\max_{0\le x\le L}|M(x)|\right) \le M_\mathrm{c} \text{ for all } \phi(x) \text{ in } \mathcal{U}(h,\widetilde{\phi})\right\} && (6)\\
&= \max\left\{h: \ \left(\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})}\ \max_{0\le x\le L}|M(x)|\right) \le M_\mathrm{c}\right\} && (7)
\end{aligned}$$

We can invert the order of the maxima inside the set.

¶ We begin by evaluating:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \max\left(\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x),\ \left|\min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x)\right|\right) \tag{8}$$

¶ To find these extrema note that:

- Other than $\phi(u)$, the integrands of both integrals in eq.(2) on p.5 have the same sign everywhere.
- Thus, extremal $M(x)$ is obtained by choosing $\phi(x) = \widetilde{\phi}(x) + h \quad \text{or} \quad \phi(x) = \widetilde{\phi}(x) - h$.
- **We consider a special case:** $\widetilde{\phi}(x)$ = positive constant.
- The results:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) \;=\; -\frac{(h-\widetilde{\phi})x(L-x)}{2} \tag{9}$$

$$\min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) \;=\; -\frac{(h+\widetilde{\phi})x(L-x)}{2} \tag{10}$$

Hence:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(h+\widetilde{\phi})x(L-x)}{2} \tag{11}$$

¶ We are now ready to evaluate the second optimization, on x, in the expression for the robustness, eq.(7) on p.5.
We find the maximum at $x = L/2$, resulting in:

$$\max_{0\le x\le L}\ \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| \;=\; \frac{(h+\widetilde{\phi})L^2}{8} \tag{12}$$

¶ The robustness is the greatest h
at which the maximum bending moment $M(x)$
does not exceed the critical value M_c.
We find:

$$\underbrace{\frac{(h+\widetilde{\phi})L^2}{8}}_{\text{max bending moment}} = \underbrace{M_c}_{\text{critical moment}} \implies \widehat{h} = \frac{8M_c}{L^2} - \widetilde{\phi} \tag{13}$$

The robust reliability $\widehat{h}$ increases as:

- The beam length L decreases.
- The nominal load $\widetilde{\phi}$ decreases.
- The critical bending moment M_c increases.

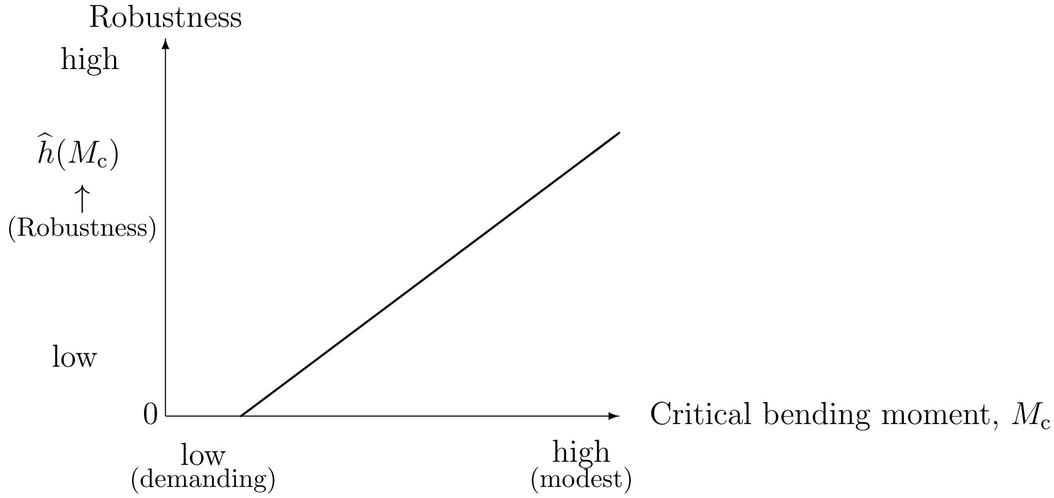

Figure 1: Robustness curve.

¶ **Two Properties:** Trade-off and zeroing.

¶ **Trade off:** robustness vs performance.
$\widehat{h}(M_c)$ *gets worse* (decreases) as M_c *gets better* (decreases).

¶ **Zeroing:** Estimated performance has zero robustness:

$$\widehat{h}(M_c) = 0 \quad \text{if} \quad M_c = \frac{\widetilde{\phi}L^2}{8} = \text{estimated bending moment} \tag{14}$$

2 Statically Loaded Beam: Continued

2.1 Load-Uncertainty Envelope

¶ **Different prior information; different uncertainty.** Examples:
- Hidden load on left half of beam.
- Flow perpendicular to beam; turbulence in middle region only.

¶ Let us now consider different prior information.
Rather than the uniform-bound info-gap model of eq.(1) on p.4,
suppose we have information which indicates that
the uncertain deviation $\phi(x) - \widetilde{\phi}(x)$ varies within an envelope:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \leq h\psi(x) \right\}, \quad h \geq 0 \tag{15}$$

where we **know:**
$\widetilde{\phi}(x)$ = nominal load profile.
$\psi(x)$ = load-uncertainty envelope.
and we **do not know:**
$\phi(x)$ = actual load profile.
h = uncertainty parameter, horizon of uncertainty.

¶ **Examples of envelope function,** $\psi(x)$:
- Hidden load on left half of beam.

$$\psi(x) = \begin{cases} 1, & 0 \leq x \leq L/2 \\ 0, & L/2 < x \leq L \end{cases} \tag{16}$$

- Flow perpendicular to beam; turbulence in middle region only.

$$\psi(x) = \sin \frac{\pi x}{L} \tag{17}$$

¶ As usual with an info-gap model, there are two levels of uncertainty:
- Unknown realization $\phi(x)$ at info-gap h.
- Unknown horizon of uncertainty, h.

¶ As before:
- The system model is eq.(2) on p.5.
- The failure criterion is eq.(3) on p.5.

¶ To find the maximum absolute bending moment
we evaluate the max and the min of $M_\phi(x)$.
The max (least negative) is obtained with the lowest possible load profile,
while
The min (most negative) is obtained with the greatest possible load profile.
We find:

$$\begin{aligned} M_1(x) &= \min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) && (18)\\ &= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u)+h\psi(u)\right] u\,\mathrm{d}u \\ &\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u)+h\psi(u)\right](L-u)\,\mathrm{d}u && (19)\\ M_2(x) &= \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) && (20)\\ &= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u)-h\psi(u)\right] u\,\mathrm{d}u \\ &\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u)-h\psi(u)\right](L-u)\,\mathrm{d}u && (21)\end{aligned}$$

We can express these succintly as:

$$\begin{aligned} M_1(x) &= \eta_1(x)+h\eta_2(x) && (22)\\ M_2(x) &= \eta_1(x)-h\eta_2(x) && (23)\end{aligned}$$

where $\eta_1(x)$ and $\eta_2(x)$ are defined implicitly in eqs.(19) and (21).

¶ Let us consider a **special case:**

The nominal load increases towards the center of the beam:

$$\widetilde{\phi}(x)=\widetilde{\phi}\sin\frac{\pi x}{L} \qquad (24)$$

where $\widetilde{\phi}$ is a known positive constant.

The uncertainty in the load increases towards the center of the beam:

$$\psi(x)=\sin\frac{\pi x}{L} \qquad (25)$$

¶ Note that $\phi(x)$, $\widetilde{\phi}(x)$ and h all have the same units.

The η-functions become:

$$\begin{aligned} \eta_1(x) &= -\frac{L^2\widetilde{\phi}}{\pi^2}\sin\frac{\pi x}{L} && (26)\\ \eta_2(x) &= \eta_1(x)/\widetilde{\phi} && (27)\end{aligned}$$

¶ The least and greatest bending moments at point x are:

$$M_1(x) = -(\widetilde{\phi}+h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{28}$$

$$M_2(x) = -(\widetilde{\phi}-h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{29}$$

¶ From this we find that the greatest absolute bending moment occurs at the midpoint of the beam:

$$\max_{0\le x\le L}\ \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(\widetilde{\phi}+h)L^2}{\pi^2} \tag{30}$$

¶ To find the robustness, we equate the maximum bending moment to the critical moment and solve for h:

$$\frac{(\widetilde{\phi}+h)L^2}{\pi^2} = M_{\mathrm{c}} \quad\Longrightarrow\quad \widehat{h} = \frac{\pi^2 M_{\mathrm{c}}}{L^2} - \widetilde{\phi} \tag{31}$$

This is quite similar to the uniform-bound case, eq.(13) on p.7.

¶ Why is:

$$\widehat{h}_{\text{env}} > \widehat{h}_{\text{uni}} \quad ? \tag{32}$$

Because:

$$\mathcal{U}_{\text{env}}(h, \widetilde{\phi}) \ \subset \ \mathcal{U}_{\text{uni}}(h, \widetilde{\phi}) \tag{33}$$

- Tighter bound on the uncertainty
  implies greatest robustness to unknown variation in the load.
- Our prior information has not substantially altered our analysis.

¶ The two info-gap models we have studied are:

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x): \ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\right\}, \quad h \ge 0 \tag{34}$$

(Eq.(1) on p. 4.)

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x): \ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\psi(x)\right\}, \quad h \ge 0 \tag{35}$$

(Eq.(15) on p. 8.)

• Both of these uncertainty models entail **unbounded rate of variation.**

• We sometimes have information which constrains the rate of variation of the uncertain function. We will now develop the tools needed to exploit this information.

2.2 Fourier Representation of a Function

¶ We interrupt our study of this example to briefly introduce the Fourier representation of a function.
We will use Fourier representations in a new type of info-gap model.

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $-L \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} \left[b_n \sin \frac{n\pi x}{L} + c_n \cos \frac{n\pi x}{L} \right] \tag{36}$$

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $0 \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \tag{37}$$

¶ How to choose the Fourier coefficients $c_0, c_1, \ldots$ in eq.(37)?
Exploit orthogonality:

$$\int_0^{\pi} \cos mx \cos nx \, \mathrm{d}x = \left\{ \begin{array}{ll} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{array} \right. \tag{38}$$

To do this, multiply both sides of eq.(37) by $\cos \frac{k\pi x}{L}$ and integrate from 0 to L:

$$\int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x = \sum_{n=0}^{\infty} c_n \int_0^L \cos \frac{k\pi x}{L} \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{39}$$

$$= \frac{c_k L}{2} \tag{40}$$

So, if we know the function $\phi(x)$ we can calculate the Fourier coefficients of its expansion:

$$c_k = \frac{2}{L} \int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x \tag{41}$$

¶ These Fourier coefficients have many interesting and important properties. First of all, they minimize the mean squared error between $\phi(x)$ and its expansion. That is, the c_n minimize:

$$S^2 = \int_0^L \left(\phi(x) - \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \right)^2 \mathrm{d}x \tag{42}$$

In fact,

$$\lim_{N \to \infty} S^2 = 0 \tag{43}$$

Another important property relates to truncated expansions:

$$\phi(x) = \sum_{n=0}^{N} c_n \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{44}$$

Regardless of the order of the expansion, N:
• Orthogonality yields the same Fourier coefficients, c_k.
• These coefficients minimize the mean squared error of the truncated expansion.

¶ Band-limited function:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \cos\frac{n\pi x}{L} \tag{45}$$

$$= c^T \gamma(x) \tag{46}$$

¶ Uncertainty in $\phi(x)$ is represented as uncertainty in Fourier coefficients c.
- For instance: c in ellipsoid of known shape and unknown size:

$$\mathcal{U}(h, \widetilde{c}) = \left\{ \phi(x) = c^T \gamma(x) : \ (c - \widetilde{c})^T W (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{47}$$

¶ Example: ps1 #4.

2.3 Geometry of Ellipsoids

¶ We need one more digression before we proceed with our example:
Geometry of ellipsoids.
The question we study in this subsection is:
What are the **directions and lengths**
of the principal axes of an ellipsoid?

¶ If: c is an N-vector and
W is a real, symmetric, positive definite matrix,
then an ellipsoid of c-vectors of dimension N is defined by:

$$c^T W c = r^2 \tag{48}$$

where r is any positive real number.

¶ Simple examples:

$$r^2 = c_1^2 w_1 + c_2^2 w_2, \quad W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \tag{49}$$

$$r^2 = c_1^2 w_1 + 2c_1 c_2 w_{12} + c_2^2 w_2, \quad W = \begin{pmatrix} w_1 & w_{12} \\ w_{12} & w_2 \end{pmatrix} \tag{50}$$

¶ To answer our question, we must solve an optimization problem.
We must find vectors c which have two properties:
- Length is extremal.
- Lie on the boundary of the ellipsoid.

¶ To optimize the length of c, it is sufficient
to optimize the square of the length of c.
So we must optimize:

$$c^T c \tag{51}$$

Let's try differential calculus:

$$0 = \frac{\mathrm{d} c^T c}{\mathrm{d} c} = 2c \quad \Longrightarrow \quad c = 0 \tag{52}$$

That's the minimum. What's the maximum? $c^T c$ is unbounded. We need the constraint.

¶ To solve this problem we will use the method of
Lagrange multipliers.

¶ A c-vector lies on the ellipsoid if eq.(48) is satisfied.
Expressing this slightly differently, the constraint on c is:

$$r^2 - c^T W c = 0 \tag{53}$$

¶ Define the objective function:

$$H = c^T c - \lambda(r^2 - c^T W c) \tag{54}$$

If we find all c-vectors which
optimize H subject to the constraint,
we will have solved the problem.

¶ Condition for extremum of H:

$$\frac{\partial H}{\partial c} = 2c - 2\lambda W c \tag{55}$$

$$\Longrightarrow \quad (I - \lambda W)c = 0 \tag{56}$$

which means that:
$c =$ is an eigenvector of W.
$\frac{1}{\lambda}$ = the corresponding eigenvalue.

¶ Define the eigenvalues and orthonormal eigenvectors of W:

$$W v_i = \mu_i v_i, \quad i = 1, \ldots, N \tag{57}$$

where:

$$0 < \mu_1 \le \cdots \le \mu_N \quad \text{and} \quad v_m^T v_n = \delta_{mn} \tag{58}$$

where δ_{mn} is the Kronecker delta function:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases} \tag{59}$$

¶ Now, since c must be an eigenvector of W, we know that:

$$c = h v_i \tag{60}$$

for some non-zero h and for any $i = 1, \ldots, N$.

Hence the constraint on c is:

$$r^2 = c^T W c = h^2 v_i^T W v_i = h^2 \mu_i \quad \Longrightarrow \quad h = \pm \frac{r}{\sqrt{\mu_i}} \tag{61}$$

¶ Thus the optimizing c-vectors are:

$$c = \pm \frac{r}{\sqrt{\mu_i}} v_i, \quad i = 1, \ldots, N \tag{62}$$

From this we see that:
The **directions** of the principal semi-axes are:

$$\pm v_1, \ldots, \pm v_N \tag{63}$$

The **lengths** of the principal semi-axes are:

$$\frac{r}{\sqrt{\mu_1}}, \ldots, \frac{r}{\sqrt{\mu_N}} \tag{64}$$

¶ Example: ps1 #5.

2.4 Fourier Ellipsoid Bounded Uncertain Load

Based on *Robust Reliability in the Mechanical Sciences,* section 3.2.4.

¶ We now consider a different type of prior information about the uncertain load profile $\phi(x)$.

¶ About $\phi(x)$ we **know:**
- Load vanishes at ends: $\phi(0) = \phi(L) = 0$.
- $\phi(x)$ is constrained to specific known spatial frequencies.
- The amplitudes of these frequencies are bounded by an ellipsoid of known shape.

¶ About $\phi(x)$ we **do not know:**
- The precise amplitudes of the Fourier coefficients.
- The size of the ellipsoid.

¶ In other words, a load profile is represented by:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \sin \frac{n\pi x}{L} \tag{65}$$

$$= c^T \sigma(x) \tag{66}$$

where:
$c =$ vector of unknown Fourier coefficients.
$\sigma(x) =$ vector of known corresponding sine functions.

¶ The uncertainty in $\phi(x)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$\mathcal{U}(h, 0) = \left\{ \phi(x) = c^T \sigma : \ c^T W c \le h^2 \right\}, \quad h \ge 0 \tag{67}$$

where W is a known, real, symmetric, positive definite matrix.[1]

¶ The system model is obtained by combining eq.(2) on p.5 for the bending moment with eq.(66):

$$M(x) = c^T \underbrace{\left[-\frac{L-x}{L} \int_0^x u\sigma(u)\, \mathrm{d}u - \frac{x}{L} \int_x^L (L-u)\sigma(u)\, \mathrm{d}u \right]}_{\zeta(x)} \tag{68}$$

$$= c^T \zeta(x) \tag{69}$$

¶ As before, failure occurs if the bending moment exceeds a critical value, as expressed in eq.(3) on p.5.

[1] For an example of a Fourier ellipsoid model see: Yakov Ben-Haim and Isaac Elishakoff, Non-Probabilistic models of uncertainty in the non-linear buckling of shells with general imperfections: Theoretical estimates of the knockdown factor. *A.S.M.E. Journal of Applied Mechanics*, Vol. 56, pp 403–410, 1989.

¶ In order to find the robustness, we must solve the following optimization:

$$\max M(x) \quad \text{for} \quad c^T W c \le h^2 \tag{70}$$

which is equivalent to:

$$\max c^T \zeta \quad \text{for} \quad c^T W c \le h^2 \tag{71}$$

To do this we employ the Cauchy inequality:

$$\left(x^T y\right)^2 \le \left(x^T x\right)\left(y^T y\right) \tag{72}$$

with equality iff:

$$x \propto y \tag{73}$$

Let us write:

$$c^T \zeta = \left(W^{1/2} c\right)^T \left(W^{-1/2} \zeta\right) \tag{74}$$

Applying Cauchy's inequality to the expression on the right:

$$\begin{aligned} \left(c^T \zeta\right)^2 &\le \left[\left(W^{1/2} c\right)^T \left(W^{1/2} c\right)\right] \left[\left(W^{-1/2} \zeta\right)^T \left(W^{-1/2} \zeta\right)\right] && (75) \\ &= \underbrace{\left[c^T W c\right]}_{\le h^2} \left[\zeta^T W^{-1} \zeta\right] && (76) \end{aligned}$$

From this we conclude that:

$$\max_{c \in \mathcal{U}(h,0)} M(x) = h \sqrt{\zeta(x)^T W^{-1} \zeta(x)} \tag{77}$$

¶ We can now express the robustness as the greatest value of the uncertainty parameter h at which the bending moment does not exceed the critical value. We find:

$$\widehat{h} = \frac{M_{\mathrm{c}}}{\max_{0 \le x \le L} \sqrt{\zeta(x)^T W^{-1} \zeta(x)}} \tag{78}$$

¶ Let us consider a **special case:**
W is the identity matrix, so
The uncertainty ellipsoid is a sphere.

¶ Now $\zeta^T W \zeta$ becomes:

$$\zeta^T(x) \zeta(x) = \frac{L^4}{\pi^4} \sum_{n=n_1}^{n_2} \frac{1}{n^4} \sin^2 \frac{n \pi x}{L} \tag{79}$$

The terms in this sum decrease rapidly with n.
Hence the maximum is dominated by the first term:

$$\begin{aligned} \max_{0 \le x \le L} \sqrt{\zeta(x)^T \zeta(x)} &\approx \max_{0 \le x \le L} \sqrt{\frac{L^4}{\pi^4} \frac{1}{n_1^4} \sin^2 \frac{n_1 \pi x}{L}} && (80) \\ &= \frac{L^2}{n_1^2 \pi^2} && (81) \end{aligned}$$

From eq.(78) we find the robustness to be:

$$\widehat{h} \approx \frac{n_1^2 \pi^2 M_{\rm c}}{L^2} \tag{82}$$

¶ Comparing this with
the robustness for the uniform-bound info-gap model,
with $\widetilde{\phi} = 0$, eq.(13) on p.7:

$$\widehat{h} = \frac{8 M_{\rm c}}{L^2} \tag{83}$$

we see that the reliability is substantially enhanced
by constraining the spatial modes of the load function.

3 Two Faces of Uncertainty

¶ Uncertainty has two faces:

- Pernicious: threatening failure, entailing risk.
- Propitious: promising windfall, sweeping reward.

¶ In making decisions we wish to:

- protect against pernicious uncertainty,

and

- facilitate propitious uncertainty.

¶ In evaluating decisions under uncertainty we wish to assess:

- risks

and

- opportunities.

¶ This we do with 2 immunity functions *(funkziot amidut)*:

- Robustness function *(funkziat hasinut)*:
  immunity against failure.
- Opportuneness function *(funkziat hizdamnut)*:
  immunity against windfall.

4 Robustness and Opportuneness: A First Look

(IGDT, section 3.1.1)

¶ Recall that an info-gap model is a **family:**

$$\mathcal{U}(h, \widetilde{u}), \quad h \geq 0 \tag{84}$$

of **nested sets:**

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h, \widetilde{u}) \subset \mathcal{U}(h', \widetilde{u}) \tag{85}$$

Thus info-gap uncertainty increases with
increasing h.
So: h is called the **uncertainty parameter.**

¶ The **robustness function** is the
greatest level of info-gap uncertainty
at which
failure cannot occur.

The **opportuneness function** is the
least level of info-gap uncertainty
at which
sweeping success can (but does not have to) occur.

The **robustness** function addresses **pernicious** uncertainty.
The **opportuneness** function addresses **propitious** uncertainty.

¶ We can begin to quantify these
immunity functions
as follows.

¶ Let q = **decision vector,** containing:
— design parameters.
— operational options.
— time of initiation.
— etc.

¶ Let u be an uncertain vector belonging to an info-gap model:

$$\mathcal{U}(h, \widetilde{u}), \quad h \geq 0 \tag{86}$$

The **robustness function** is:

$$\widehat{h}(q) = \max\{h : \text{ minimal requirements are satisfied } \textbf{for all } u \in \mathcal{U}(h, \widetilde{u})\} \tag{87}$$

The **opportuneness function** is:

$$\widehat{\beta}(q) = \min\{h : \text{ sweeping success is enabled } \textbf{for some } u \in \mathcal{U}(h, \widetilde{u})\} \tag{88}$$

¶ $\widehat{h}(q)$ and $\widehat{\beta}(q)$ are
dual functions
or
complementary functions.

For $\widehat{h}(q)$: **bigger is better.**
For $\widehat{\beta}(q)$: **big is bad.**

¶ $\widehat{h}(q)$ entails a **maximization:**
Not of performance or outcome of decision.
Rather: ∘ Immunity to uncertainty is maximized.
∘ Performanced is **satisficed.**

¶ To **satisfice** (OED):
"To decide on and pursue a course of action
that will satisfy the minimal requirements
needed to achieve a particular goal."
(Herb Simon, psychologist and economist.)

¶ $\widehat{\beta}(q)$ entails a **minimization:**
Not of damage resulting from unknown events.
Rather: minimize level of uncertainty needed to
enable **windfall.**

¶ We can define **windfalling** as:
To decide on and pursue a course of action that will
minimize the immunity to propitious uncertainty
in an attempt to enable highly ambitious goals.

5 Immunity Functions

(IGDT, Section 3.1.2)

¶ Often the success of a decision is expressed by a
scalar **reward function** *(funkziat toelet)*: $R(q, u)$
which depends on:
$q =$ decision vector.
$u =$ uncertain vector in an info-gap model.
E.g. $R(q, u) =$
- Degree of stability.
- Rate of mixing.
- Duration of life.
- Profit.

For all these entities a **large value** if $R(q, u)$ is desirable.

¶ Given a reward function, $R(q, u)$, the
minimal requirement in eq.(87) on p.22 is:
$R(q, u) \geq r_{\text{c}}$
where $r_{\text{c}} =$ critical, survival level of reward.

Likewise, the **sweeping success** in eq.(88) on p.22 is:
$R(q, u) \geq r_{\text{w}}$
where $r_{\text{w}} =$ windfall reward.
and
$r_{\text{w}} \gg r_{\text{c}}$.

¶ We can now define $\widehat{h}$ and $\widehat{\beta}$ more precisely.

¶ The **robustness function** is:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \ \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm c} \right\} \tag{89}$$

We can analyze this as follows:

$$\widehat{h}(q, r_{\rm c}) = \underbrace{\max \left\{ h : \right.}_{\substack{\text{max} \\ \text{uncertainty} \\ h \text{ so that}}} \quad \underbrace{\min_{u \in \mathcal{U}(h, \widetilde{u})}}_{\substack{\text{all } u \text{ up to} \\ \text{uncertainty} \\ h \text{ guarantee}}} \quad \underbrace{R(q, u) \geq r_{\rm c}}_{\substack{\text{minimal} \\ \text{requirement} \\ \text{for} \\ \text{survival}}} \quad \left. \right\}$$

$\widehat{h}(q, r_{\rm c})$ is the	maximum tolerable h
so that	all u up to uncertainty h
satisfy the	minimal requirement for survival.

¶ The **Opportuneness function** is:

$$\widehat{\beta}(q, r_{\rm w}) = \min\left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm w} \right\} \tag{90}$$

We can analyze this as follows:

$$\widehat{\beta}(q, r_{\rm w}) = \underbrace{\min\left\{ h : \right.}_{\substack{\text{least} \\ \text{uncertainty} \\ h \text{ so that}}} \quad \underbrace{\max_{u \in \mathcal{U}(h, \widetilde{u})}}_{\substack{\text{some } u \\ \text{up to} \\ \text{uncertainty} \\ h \text{ enables}}} \quad \underbrace{R(q, u) \geq r_{\rm w}}_{\substack{\text{sweeping} \\ \text{success} \\ \text{or windfall}}} \quad \left. \right\}$$

$\widehat{\beta}(q, r_{\rm w})$ is the	least h
so that	some u up to uncertainty h
enables the	possibility of windfall success.

¶ Note that $\widehat{h}$ and $\widehat{\beta}$ are extrema of sets of h-values.

Define the sets:

$$\mathcal{A}(q, r_{\rm c}) = \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm c} \right\} \quad (91)$$

$$\mathcal{B}(q, r_{\rm w}) = \left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm w} \right\} \quad (92)$$

Thus:

$$\widehat{h}(q, r_{\rm c}) = \text{LUB } \mathcal{A}(q, r_{\rm c}) \quad (93)$$

$$\widehat{\beta}(q, r_{\rm w}) = \text{GLB } \mathcal{B}(q, r_{\rm w}) \quad (94)$$

Also, if:

$$\mathcal{A}(q, r_{\rm c}) = \emptyset \quad (95)$$

then define:

$$\widehat{h}(q, r_{\rm c}) = 0 \quad (96)$$

because eq.(95) implies:

- No immunity to failure.
- Infinitestimal variation entails possibility of failure.

Likewise, if:

$$\mathcal{B}(q, r_{\rm w}) = \emptyset \tag{97}$$

then define:

$$\widehat{\beta}(q, r_{\rm w}) = \infty \tag{98}$$

because eq.(97) implies:

- No value of h is large enough to enable windfall $r_{\rm w}$.
- The immunity to windfall is unbounded.

¶ Up to now we have considered
reward functions $R(q, u)$ for which
large reward is desirable.

¶ In some situations, **small** $R(q, u)$ is preferred over
large $R(q, u)$.
E.g. $R(q, u)$ is measure of
- **in**stability of the system.
- Financial **loss.**
- **Delay** in implementation.

¶ If small $R(q, u)$ is preferred over large $R(q, u)$
then we define the immunity functions:

$$\widehat{h}(q, r_{\rm c}) = \max\left\{h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \le r_{\rm c}\right\} \quad (99)$$

$$\widehat{\beta}(q, r_{\rm w}) = \min\left\{h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \le r_{\rm w}\right\} \quad (100)$$

where:

$$r_{\rm w} \ll r_{\rm c} \quad (101)$$

¶ Note that in both formulations,
- eqs.(89) and (90), (pp.25, 26)
- eqs.(99) and (100), (p.29)

"Bigger is better" for $\widehat{h}(q, r_{\rm c})$
"Big is bad" for $\widehat{\beta}(q, r_{\rm w})$

6 Design of a Vibrating Cantilever

(IGDT, sec. 3.3.1)

6.1 Design Problem

¶ We now consider an example:
Vibration control in a cantilever
subject to uncertain dynamic excitation.

¶ The cantilever: rigid beam which is clamped at one end.
See transparency of:
- Galileo's cantilever.
- Atomic force microscope.

¶ The cantilever is the paradigm for:
- Tall building.
- Radio tower.
- Crane (agoran).
- Airplane wing.
- Turbine blade.
- Diving board.
- Canon barrel.
- Atomic force microscope.
- etc.

¶ Central goal in design of the cantilever:
Control of vibration resulting from external loads.

¶ Two basic approaches:
1. Prevent vibration by stiffening the beam.
2. Absorb vibration by dissipating energy.

¶ These design concepts are **not** mutually exclusive.
They can be implemented together.

¶ These design concepts are relevant in different circumstances
as we will see.

6.2 Robustness Function

¶ We will use the **robustness function**
to evaluate the design options.

¶ Later we will consider the **opportuneness function.**

¶ As usual, the three components of the analysis are:
1. System model.
2. Failure (or performance) criterion.
3. Uncertainty model.

¶ We use a simple **system model:**
Rigid vibration around the clamped base.
$\theta(t)$ = angle of deflection of beam.
$u(t)$ = moment of force at base.
Equation of motion:

$$J\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k\theta = u(t) \tag{102}$$

J = moment of inertia of beam wrt rotation at base.
c = damping coefficient.
k = rotational stiffness coefficient.

¶ Solution of eq. of motion, for:
- Zero initial conditions, $\theta(0) = \dot{\theta}(0) = 0$
- Subcritical damping, $\zeta^2 < 1$:

$$\theta_u(t) = \int_0^t u(\tau) f(t-\tau)\,\mathrm{d}\tau \tag{103}$$

$f(t)$ = impulse response function:

$$f(t) = \frac{1}{J\omega_{\mathrm{d}}}\mathrm{e}^{-\zeta\omega t}\sin\omega_{\mathrm{d}}t \tag{104}$$

$\omega^2 = k/J$ = squared natural frequency.
$\zeta = \frac{c}{2J\omega}$ = dimensionless damping coefficient.
$\omega_{\mathrm{d}} = \omega\sqrt{1-\zeta^2}$ = damped natural frequency.

¶ We now consider the **uncertainty model.**
What we **know** about the load is:

- The nominal load, $\widetilde{u}(t)$.
- The actual loads are transient:
  - May vary rapidly,
  - May attain large deviations from the nominal load.

We will model load uncertainty with the
cumulative energy bound info-gap model:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{105}$$

¶ The **performance criterion:**
Deflection must not exceed critical value:

$$|\theta(t)| \le \theta_{\mathrm{c}} \tag{106}$$

In terms of reward functions, define:

$$R(q, u) = |\theta(t)| \tag{107}$$

$u =$ uncertain load.
$q =$ design concept, as expressed in damping c and stiffness k.

¶ The robustness function can be defined as in eq.(99) on p.29:

$$\widehat{h}(q, \theta_{\mathrm{c}}) = \max \left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| \le \theta_{\mathrm{c}} \right\} \tag{108}$$

$\widehat{h}(q, \theta_{\mathrm{c}})$ is the maximum tolerable info-gap.

¶ We now evaluate:

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| \tag{109}$$

¶ Note that $\theta_u(t)$ in eq.(103) on p.33 can be re-written:

$$\theta_u(t) = \int_0^t u(\tau) f(t - \tau) \, \mathrm{d}\tau \tag{110}$$

$$= \int_0^t [u(\tau) - \widetilde{u}(\tau)] \, f(t - \tau) \, \mathrm{d}\tau + \underbrace{\int_0^t \widetilde{u}(\tau) f(t - \tau) \, \mathrm{d}\tau}_{\widetilde{\theta}(t)} \tag{111}$$

where $\widetilde{\theta}(t) =$ nominal deflection.

We need the Schwarz inequality:

$$\left(\int_a^b f(t) g(t) \, \mathrm{d}t \right)^2 \le \int_a^b f(t)^2 \, \mathrm{d}t \int_a^b g(t)^2 \, \mathrm{d}t \tag{112}$$

with equality iff:

$$f(t) = c g(t) \tag{113}$$

for some non-zero constant c.

Now notice that the first integral in eq.(111) on p.34 is bounded:

$$\left(\int_0^t [u(\tau) - \widetilde{u}(\tau)] \, f(t-\tau) \, \mathrm{d}\tau \right)^2 \le \underbrace{\left(\int_0^t [u(\tau) - \widetilde{u}(\tau)]^2 \, \mathrm{d}\tau \right)}_{\mathrm{I}} \underbrace{\left(\int_0^t f^2(t-\tau) \, \mathrm{d}\tau \right)}_{\mathrm{II}} \tag{114}$$

Note:
- From the info-gap model we know that Integral I $\le h^2$.
- Integral II is known.
- The info-gap model allows us to choose $u(\tau) - \widetilde{u}(\tau) \propto f(t-\tau)$.
- Thus, from eqs.(111) and (114):

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| = h \sqrt{\int_0^t f^2(\tau) \, \mathrm{d}\tau} + \left| \widetilde{\theta}(t) \right| \tag{115}$$

¶ We can now express the robustness function:
- Equate $\max|\theta_u(t)|$ to $\theta_{\rm c}$.
- Solve for h, yielding $\widehat{h}$:

$$h\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau} + \left|\widetilde{\theta}(t)\right| = \theta_{\rm c} \quad \Longrightarrow \quad \widehat{h}(q, \theta_{\rm c}) = \frac{\theta_{\rm c} - \left|\widetilde{\theta}(t)\right|}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{116}$$

unless this is negative, in which case $\widehat{h} = 0$.

6.3 Numerical Example

¶ We will consider a specific example.
Nominal input $\widetilde{u}(t)$ is square:

$$\widetilde{u}(t) = \begin{cases} \widetilde{u}_o, & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{117}$$

The nominal response can be calculated:

$$\widetilde{\theta}(t) = \theta_{\widetilde{u}}(t) = \frac{(1-\zeta^2)\widetilde{u}_o}{J\omega_{\rm d}}\gamma(t) \tag{118}$$

where $\gamma(t)$ is a known function.

For notational convenience we represent
integral II in eq.(114) on p.35 as:

$$\sqrt{\int_0^t f^2(t-\tau)\,{\rm d}\tau} = \frac{1-\zeta^2}{2J\omega_{\rm d}^{3/2}}\phi(t) \tag{119}$$

where $\phi(t)$ is a known function.

Now the robustness function can be expressed:

$$\widehat{h}(q, \theta_{\rm c}) = \frac{2J\theta_{\rm c}\omega^2\sqrt{\omega_{\rm d}} - 2\sqrt{\omega_{\rm d}}|\widetilde{u}_o\gamma(t)|}{\omega\phi(t)} \tag{120}$$

Recall: q = decision vector = (c, k).
which is embedded in ω and $\omega_{\rm d}$.

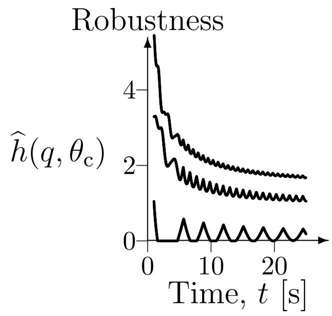

Figure 2: Robustness versus time for three values of the natural frequency $\omega = 1$, 3 and 4 (bottom to top). Negligible damping: $\zeta = 0.01$. $1 = J\theta_c = \widetilde{u}_0$. $T = 5$.

¶ $\widehat{h}(q,\theta_c)$ vs. t is plotted in fig. 2
For various natural frequencies: $\omega = 1$, 3 and 4 (bottom to top).
With negligible damping: $\zeta = 0.01$.

- $\widehat{h}$ oscillates but tends to decrease over time.
- At low stiffness ($\omega = 1$) the robustness periodically vanishes.
- At moderate and high stiffness ($\omega = 3$, 4)
  $\widehat{h}$ oscillates but does not reach zero for the duration shown.
- The transition from rapid to slow decrease in $\widehat{h}$
  occurs about at $t = T$ (end of nominal input).

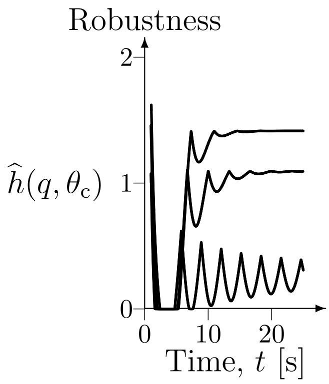

Figure 3: Robustness versus time for three values of the damping ratio $\zeta = 0.03,\ 0.3,\ 0.5$ (bottom to top). Fixed natural frequency $\omega = 1$. $1 = J\theta_c = \tilde{u}_0$. $T = 5$.

¶ Now consider fig. 3, which shows
$\widehat{h}(q, \theta_c)$ vs. t for various damping ratios:
$\zeta = 0.03$, 0.3 and 0.5
at low stiffness: $\omega = 1$.

- Lowest curve is quite similar to lowest curve in fig. 2.
- With large damping ($\zeta = 0.3$ or 0.5):
  $\widehat{h}$ is small for $t \le T$
  $\widehat{h}$ is large and nearly constant thereafter.

¶ Comparing figs. 2 and 3:

- Fig. 2 is based on "stiffness" design concept, with negligible damping.
- Fig. 3 is based on "dissipation" design concept, with negligible stiffness.
- The choice of a design concept depends on the time frame of interest:
  - $t < T$ calls for "stiffness" design.
  - $t > T$ calls for "dissipation" design.
  - $t > 0$ calls for combined "stiffness" and "dissipation" design.

6.4 Opportuneness Function

¶ We now consider the opportuneness function.
Windfall reward: angular deflection $\theta_{\rm w}$
much less than the survival requirement, $\theta_{\rm c}$:

$$\theta_{\rm w} < \widetilde{\theta} < \theta_{\rm c} \tag{121}$$

¶ Immunity to windfall, $\widehat{\beta}(q, \theta_{\rm w})$: the **least** info-gap at which windfall is **possible.**

¶ Analogous to eq.(108) on p. 34:

$$\widehat{\beta}(q, \theta_{\rm w}) = \min \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| \le \theta_{\rm w} \right\} \tag{122}$$

¶ Proceeding as in eq.(115) on p. 35 we find:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| = -h \sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| \tag{123}$$

Equating this to $\theta_{\rm w}$ and solving for h yields the opportuneness function, as in eq.(116) on p. 36:

$$-h \sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| = \theta_{\rm w} \quad \Longrightarrow \quad \widehat{\beta}(q, \theta_{\rm w}) = \frac{\left| \widetilde{\theta}(t) \right| - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau}} \tag{124}$$

unless this is negative, in which case $\widehat{\beta} = 0$.

Why does $\widehat{\beta} = 0$ in this case?

$\widehat{\beta} < 0$ only if $\left| \widetilde{\theta}(t) \right| < \theta_{\rm w}$.
This means that the **nominal response** $|\widetilde{\theta}(t)|$
is less than the **windfall response** $\theta_{\rm w}$.
Hence windfall is possible even without uncertainty:
The immunity to windfall is zero.

¶ Compare $\widehat{\beta}(q, \theta_{\rm w})$ to the robustness in eq.(116) on p. 36:

$$\widehat{h}(q, \theta_{\rm c}) = \frac{\theta_{\rm c} - \left|\widetilde{\theta}(t)\right|}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{125}$$

We see that the immunity functions are related as:

$$\widehat{\beta}(q, \theta_{\rm w}) = -\widehat{h}(q, \theta_{\rm c}) + \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{126}$$

¶ We now consider **antagonism** and **sympathy** of the immunity functions.

¶ The immunity functions $\widehat{\beta}(q, \theta_{\rm w})$ and $\widehat{h}(q, \theta_{\rm c})$ are **sympathetic** if they can be improved simultaneously. They are **antagonistic** if either can be improved only at the expense of the other.

¶ For example, we can vary ω.
The immunity functions are **antagonistic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} > 0}_{\text{improving with } \omega} \qquad \text{and} \qquad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} > 0}_{\text{degenerating with } \omega} \tag{127}$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} < 0}_{\text{degenerating with } \omega} \qquad \text{and} \qquad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0}_{\text{improving with } \omega} \tag{128}$$

¶ On the other hand,
the immunity functions are **sympathetic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} > 0}_{\text{improving with } \omega} \qquad \text{and} \qquad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0}_{\text{improving with } \omega} \tag{129}$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} < 0}_{\text{degenerating with } \omega} \qquad \text{and} \qquad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} > 0}_{\text{degenerating with } \omega} \tag{130}$$

¶ In short, the immunity functions are
sympathetic wrt ω if and only if:

$$\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} \frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0 \tag{131}$$

¶ Return to eq.(126) on p. 42.

- Question: Under what conditions will $\widehat{h}$ and $\widehat{\beta}$ always be sympathetic?
- Answer: If and only if their optima coincide. See fig. 4.

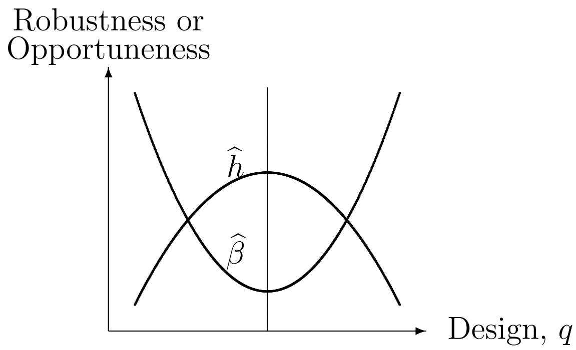

Figure 4: Sympathetic robustness and opportuneness curves.

¶ When will this occur? Iff

$$\frac{\partial \widehat{\beta}}{\partial q} = 0 = \frac{\partial \widehat{h}}{\partial q} \tag{132}$$

From eq.(126) we see that this will happen only if, at the same q, we also have:

$$\frac{\partial D}{\partial q} = 0 \tag{133}$$

where we define:

$$D = \frac{\theta_{\mathrm{c}} - \theta_{\mathrm{w}}}{\sqrt{\int_0^t f^2(\tau)\, \mathrm{d}\tau}} \tag{134}$$

"Usually" this will not happen, which means that, instead of fig. 4, we will have fig. 5.

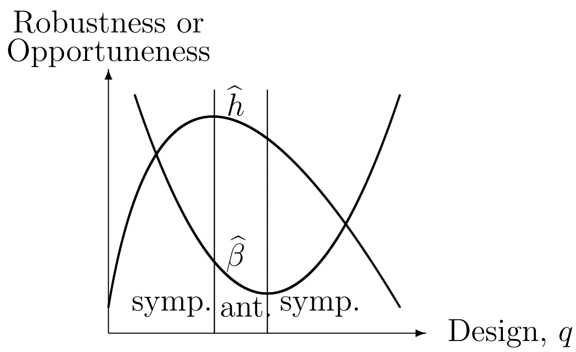

Figure 5: Robustness and opportuneness curves which are both sympathetic and antagonistic.

7 Generic Decision Algorithms

¶ We have defined the immunity functions:
$\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$
on the basis of:
- an info-gap model of uncertainty, $\mathcal{U}(h, \widetilde{u})$, $h \geq 0$.
- a scalar reward function, $R(q, u)$.

We will now show that $\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$
can be defined with a:
generic decision algorithm.

¶ $D(q, u)$ = generic decision algorithm
whose value is the "answer" or "response" to
the "input" $u \in \mathcal{U}(h, \widetilde{u})$ for some h.
$q =$ decision vector specifying the structure of D.

¶ Decisions may be an inference about a system, e.g.:
- Is it safe? Yes or no.
- Is the max response $\leq$ a critical value?

Or the decision algorithm may:
- Select one from among several hypotheses about the system or environment.
- Select one from among several design options.
- Select one from among several operational alternatives.

¶ The robustness of a decision algorithm can be formulated in several different ways.

¶ One possibility:

$$
\begin{aligned}
\widehat{h}(q) &= \text{greatest info-gap uncertainty such that} \\
&\quad \text{the } \textbf{actual design} = \text{the } \textbf{nominal design.} && (135) \\
&= \text{max info-gap at which } D(q,u) \text{ is stable.} && (136) \\
&= \max\{h:\ D(q,u) = D(q,\widetilde{u}) \quad \text{for all} \quad u \in \mathcal{U}(h,\widetilde{u})\} \\
&= \text{max info-gap at which} && (137) \\
&\quad \text{the } \textbf{best available decision } D(q,\widetilde{u}) \\
&\quad \text{is the same as} \\
&\quad \text{the } \textbf{most realistic decision } D(q,u). && (138)
\end{aligned}
$$

¶ An immediate extension:

$$
\begin{aligned}
\widehat{h}(q) &= \max\{h:\ \|D(q,u) - D(q,\widetilde{u})\| \le r_{\mathrm{c}}\ \forall\ u \in \mathcal{U}(h,\widetilde{u})\} && (139) \\
&= \text{max info-gap at which} \\
&\quad D(q,\widetilde{u}) \textbf{ errs no more} \text{ than } r_{\mathrm{c}}. && (140) \\
&= \text{max info-gap at which} \\
&\quad D(q,u) \textbf{ dithers no more} \text{ than } r_{\mathrm{c}}. && (141)
\end{aligned}
$$

¶ Let us identify when **decision robustness** $\widehat{h}(q, r_\mathrm{c})$
is a relevant measure of **correctness** or **validity**
of the decision itself.
The discussion has 3 parts.

1. We assume that
$\mathcal{U}(h, \widetilde{u})$, $h \geq 0$, **accurately represents uncertain variation**
in the system or environment.
This means that $\mathcal{U}(h, \widetilde{u})$, $h \geq 0$ is rich enough to include,
at some h,
a realistic representation of the system or environment.

2. Hence, **large robustness** $\widehat{h}(q, r_\mathrm{c})$ means that
the **nominal decision** $D(q, \widetilde{u})$
is the same as the **true decision** $D(q, u)$
for a large range of real systems.

3. In summary:
if $\mathcal{U}(h, \widetilde{u})$ represents realistic variation
then large $\widehat{h}(q, r_\mathrm{c})$ warrants the decision $D(q, \widetilde{u})$.

¶ We can also define the opportuneness function as a generic decision:

$$\widehat{\beta}(q, r_\mathrm{w}) = \min \left\{ h : \ \| D(q, u) - D(q, \widetilde{u}) \| \leq r_\mathrm{w} \text{ for some } u \in \mathcal{U}(h, \widetilde{u}) \right\} \tag{142}$$

This is the same as the $\widehat{\beta}$ defined earlier.

8 Multi-criterion Reward

¶ In some situations there may be:
multiple relevant reward criteria or functions:
$R_i(q,u), \quad i = 1, 2, \ldots$.
Each reward function may have its own
critical threshold $r_{c,i}, \quad i = 1, 2, \ldots$.
and
windfall threshold $r_{w,i}, \quad i = 1, 2, \ldots$.
Immunity functions can be defined for each criterion:
$\widehat{h}_i(q, r_{c,i}), \quad \widehat{\beta}_i(q, r_{w,i})$.

¶ There are various ways to **combine** the immunity functions.
One combination of robustness functions is to define:

$$\widehat{h}_i(q, r_c) = \text{overall robustness.} \quad r_c = (r_{c,1}, r_{c,2}, \ldots) \tag{143}$$

$$= \text{robustness of most vulnerable criterion.} \tag{144}$$

$$= \min_i \widehat{h}_i(q, r_{c,i}) \tag{145}$$

We have used this in project management and other examples.
¶ In a similar vein a **combined** opportuneness function is:

$$\widehat{\beta}_i(q, r_w) = \text{overall opportuneness.} \quad r_w = (r_{w,1}, r_{w,2}, \ldots) \tag{146}$$

$$= \text{opportuneness of least opportune criterion.} \tag{147}$$

$$= \max_i \widehat{\beta}_i(q, r_{w,i}) \tag{148}$$

¶ There are other ways of combining multiple criteria,
some of which we will encounter.

9 Three Components of Info-gap Decision Models

¶ A decision model always has three components:
- A system model.
- A performance requirement.
- An uncertainty model.

¶ A **system model** is represented by the
reward or performance function $R(q, u)$.
This function expresses the relation between
input (from the environment, etc.)
and
output (result of action, decision, etc.).
The choice of the reward function is not unique,
but depends on the issues which are relevant.

¶ The **performance requirement** is of the form:

$$R(q, u) \geq r \quad \text{or} \quad R(q, u) \leq r.$$

where:
$r =$ critical level of reward (robust satisficing).
or
$r =$ windfall level of reward (opportune windfalling).

¶ The **uncertainty model** is an info-gap model, $\mathcal{U}(h, \widetilde{u}), \ \ h \geq 0$.
There may be more than one info-gap model.

¶ It is important to stress the role of
$q =$ decision or design vector.

10 Preferences

¶ We have noted that, for the robustness function, $\widehat{h}(q, r_{\rm c})$:
bigger is better.

- This implies that, for any two choices of the decision vector, q:
  $q \succ q'$
  if $\widehat{h}(q, r_{\rm c}) > \widehat{h}(q', r_{\rm c})$.
- This establish a **preference ordering** on decision options at specified demanded performance, $r_{\rm c}$.
- The preference orderings may be different at different $r_{\rm c}$ values.

¶ We can define a **robust-optimal decision** $\widehat{q}_{\rm c}(r_{\rm c})$:

$$\widehat{h}(\widehat{q}_{\rm c}(r_{\rm c}),\, r_{\rm c}) = \max_{q \in \mathcal{Q}} \widehat{h}(q, r_{\rm c}) \tag{149}$$

where $\mathcal{Q}$ = set of available options.
¶ Note: optimal action $\widehat{q}_{\rm c}(r_{\rm c})$ depends on demanded performance $r_{\rm c}$.

¶ Since both:

- the preference ordering, "$\succ$" and
- the optimal action $\widehat{q}_{\rm c}(r_{\rm c})$

depend on the choice of the demanded performance $r_{\rm c}$,
we see that
info-gap decision theory does **not determine**
a unique 'rational decision'.
Rather, $\widehat{h}(q, r_{\rm c})$ is a quantitative **decision support tool**
with which we evaluate and explore options.

¶ We have noted that, for the opportuneness function, $\widehat{\beta}(q, r_{\rm w})$:

big is bad.

- This implies that, for any two choices of the decision vector, q:
  $q \succ q'$
  if $\widehat{\beta}(q, r_{\rm w}) < \widehat{\beta}(q', r_{\rm w})$.
- This establish a **preference ordering** on decision options at specified windfall performance, $r_{\rm w}$.
- The preference orderings may be different at different $r_{\rm w}$ values.
- The opportuneness-windfall preference ordering may differ from the robust-satisficing preference ordering.

¶ We can define a **windfall-optimal decision** $\widehat{q}_{\rm w}(r_{\rm w})$:

$$\widehat{\beta}(\widehat{q}_{\rm w}(r_{\rm w}),\, r_{\rm w}) = \min_{q \in \mathcal{Q}} \widehat{\beta}(q, r_{\rm w}) \tag{150}$$

where $\mathcal{Q}$ = set of available options.

¶ Note: optimal action $\widehat{q}_{\rm c}(r_{\rm w})$ depends on windfall performance $r_{\rm w}$.

11 Trade-offs

¶ We use the immunity functions, $\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$,
to explore options and form preferences.
Several rather different trade-offs arise.

¶ One trade-off is robustness vs. reward:

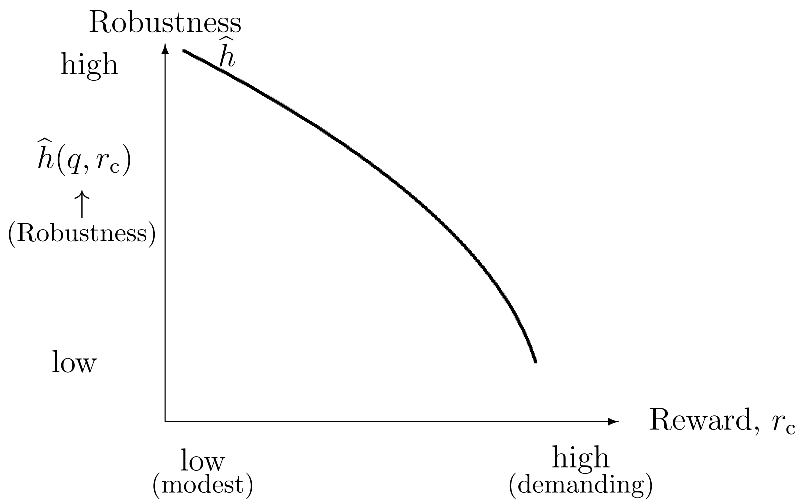

Figure 6: Robustness curve.

¶ In this figure: large r_c is better than small r_c.

- When this is true:
  The robustness vs. reward curve
  decreases monotonically with increasing critical reward.
  (As in fig. 6.)
- When small r_c is better than large r_c:
  The robustness vs. reward curve
  increases monotonically with increasing critical reward.
- The generalization:
  The robustness vs. reward curve
  decreases monotonically with increasing demanded performance.

¶ The trade-off:
High reward (great demands on performance)
is obtained in exchange for
low robustness to uncertainty.

¶ The position of the robustness curve indicates a type of **gambling.**
Consider 2 strategies whose $\widehat{h}$-functions are:

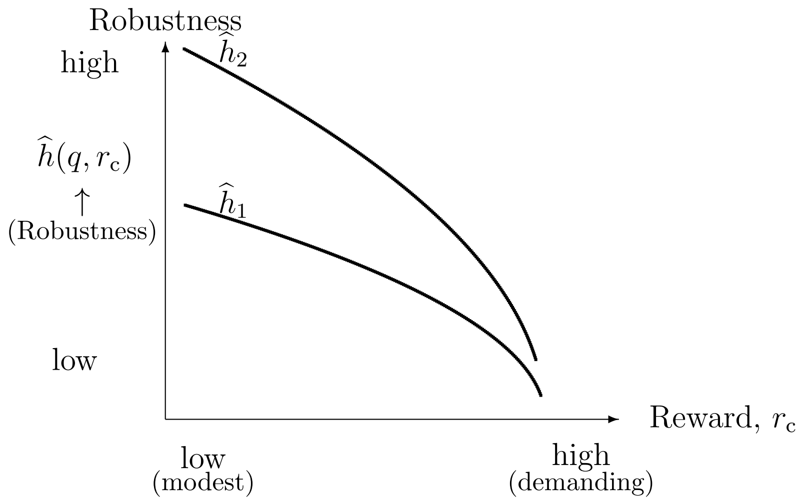

Figure 7: Robustness curve.

¶ We interpret these strategies as 'bold' and 'cautious':

- The upper strategy, $\widehat{h}_2(q, r_c)$, is **bold:**
  - At any demanded reward r_c,
    $\widehat{h}_2$ tolerates more uncertainty than $\widehat{h}_1$.
  - At any ambient uncertainty, h,
    $\widehat{h}_2$ can demand more reward than $\widehat{h}_1$.
- The upper strategy, $\widehat{h}_2(q, r_c)$, would look **risky**, **rash**,
  from the perspective of the lower strategy, $\widehat{h}_1(q, r_c)$,
  which is **cautious.**

¶ The opportuneness function also shows a trade-off:

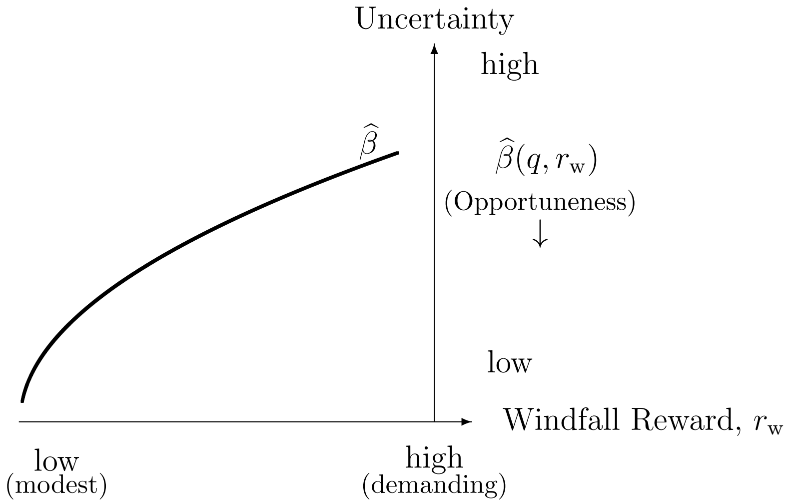

Figure 8: An opportuneness curve.

¶ The trade-off:

- High windfall reward is possible only at high ambient uncertainty.
- Low uncertainty can be bought only by giving up windfall opportunity.

¶ There is a coherence between
• robustness vs. reward trade-off
and
• certainty vs. windfall trade-off.
In both cases,
as the decision maker gives up expectation by reducing demand
(reducing r_c or r_w),
both $\widehat{h}$ and $\widehat{\beta}$ show more optimistic picture.

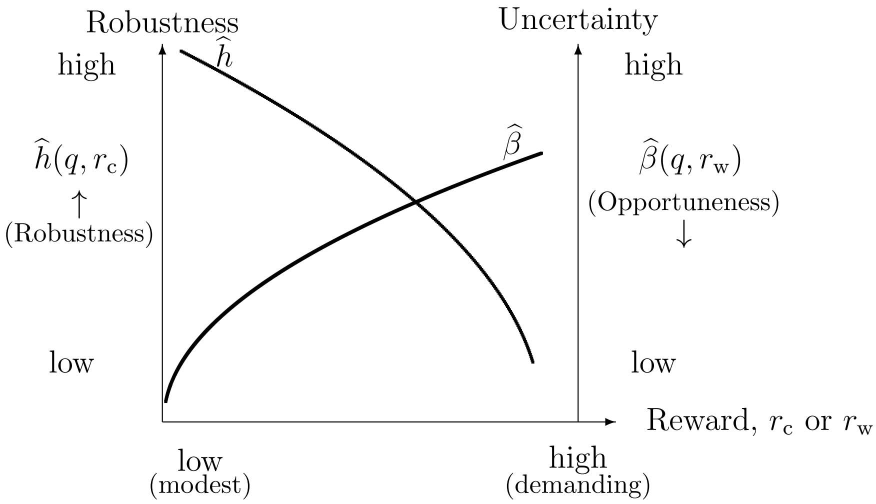

Figure 9: Robustness and opportuneness curves.

¶ Later we will explore a different type of trade-off.
We will explore the question:
• If q is changed to increase $\widehat{h}(q, r_c)$,
will $\widehat{\beta}(q, r_w)$ get better or worse?
• That is, are robustness and opportuneness
antagonistic or **sympathetic?**

12 Portfolio Investment

(IGDT, section 3.3.6)

¶ For many decision problems, the
reward R
is proportional to the
investment of resource q,
while the
coefficient of proportionality u is uncertain:

$$R(q,u) = \sum_{i=1}^{N} q_i u_i = q^T u \tag{151}$$

¶ The prototype is
portfolio investment q
with uncertain return u.
q_i = amount invested in commodity i.
u_i = dollar earned for each dollar invested in commodity i.

¶ This is also typical of many other decision problems:
- Resource distribution with proportional return.
- Elastic deflection at small strain: q_i is force, u_i is strain.
- Acoustic response.
- etc.

¶ We will consider uncertain u-vectors with the following information:
- Nominal $\widetilde{u}$ is known, calculated as historical mean.
- Shape of clusters of u-vectors is roughly known.
  We have the historical covariance of u-vectors.

Thus we will adopt an ellipsoid-bound info-gap model:

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u = \widetilde{u} + v : \ v^T W v \le h^2 \right\}, \quad h \ge 0 \tag{152}$$

where W is a known, real, symmetric, positive definite matrix, chosen as the inverse of the historical covariance matrix.

12.1 Robustness Function

¶ $\widehat{h}(q, r_c) =$ greatest uncertainty at which
reward is no less than r_c
for investment portfolio q.

$$\widehat{h}(q, r_c) = \max \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_c \right\} \tag{153}$$

To evaluate $\widehat{h}(q, r_c)$ we must determine:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = q^T \widetilde{u} + \min_{v^T W v \leq h^2} q^T v \tag{154}$$

¶ To evaluate this optimum we use **Lagrange optimization.**
Define:

$$H = q^T v + \lambda \left(h^2 - v^T W v \right) \tag{155}$$

The condition for an extremum:

$$0 = \frac{\partial H}{\partial v} = q - 2\lambda W v \tag{156}$$

$$\Longrightarrow \qquad v = \frac{1}{2\lambda} W^{-1} q \tag{157}$$

Using the constraint:

$$h^2 = v^T W v = \frac{1}{4\lambda^2} q^T W^{-1} W W^{-1} q \tag{158}$$

which leads to:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^T W^{-1} q}} \tag{159}$$

Hence:

$$v = \frac{\pm h}{\sqrt{q^T W^{-1} q}} W^{-1} q \tag{160}$$

So the minimum is:

$$\min_{v^T W v \leq h^2} q^T v = -h \sqrt{q^T W^{-1} q} \tag{161}$$

Consequently:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = q^T \widetilde{u} - h \sqrt{q^T W^{-1} q} \tag{162}$$

¶ To find $\widehat{h}$:
Equate this minimum to r_c and solve for h:

$$\widehat{h}(q, r_c) = \frac{q^T \widetilde{u} - r_c}{\sqrt{q^T W^{-1} q}} \tag{163}$$

unless this is negative, in which case:

$$\widehat{h}(q, r_c) = 0 \tag{164}$$

Note:

- Trade-off between robustness, $\widehat{h}(q, r_c)$, and satisficed return, r_c.
- Zero robustness at nominal return, $q^T \widetilde{u}$.

12.2 Robust Optimal Investment

¶ Question: how to choose the investment vector q?
Strategy:
- $\widehat{h}(q, r_c)$ depends on the decision vector q.
- For $\widehat{h}$ we know that: "bigger is better".
- So, choose q to maximize $\widehat{h}(q, r_c)$
  subject to budget constraint:

$$\sum_{i=1}^{N} q_i = Q = \text{total available budget} \tag{165}$$

$q_i > 0 \implies$ buy commodity i.
$q_i < 0 \implies$ sell commodity i.
¶ To express eq.(165) vectorially, define the N-vector:

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{166}$$

Thus:

$$\sum_{i=1}^{N} q_i = q^T \mathbf{1} \tag{167}$$

So the constraint is:

$$q^T \mathbf{1} = Q \tag{168}$$

¶ Consider a special case:

$$\widetilde{u}_i = u_o \quad \text{for all} \quad i \tag{169}$$

That is: all commodities have the same nominal value.
Of course, the uncertainties may differ between commodities.
Eq.(169) can be expressed:

$$\widetilde{u} = u_o \mathbf{1} \tag{170}$$

¶ The robustness, eq.(163), becomes:

$$\widehat{h}(q, r_c) = \frac{u_o q^T \mathbf{1} - r_c}{\sqrt{q^T W^{-1} q}} \tag{171}$$

$$= \frac{u_o Q - r_c}{\sqrt{q^T W^{-1} q}} \tag{172}$$

¶ So, how to choose the investment vector q?
From eq.(172) we maximize $\widehat{h}$
by choosing q to minimize $q^T W^{-1} q$
subject to the constraint $q^T \mathbf{1} = Q$.

¶ We again use Lagrange optimization. The optimal q is:

$$\widehat{q}_{\rm c} = \frac{Q}{\mathbf{1}^T W \mathbf{1}} W \mathbf{1} \tag{173}$$

The optimal robustness becomes:

$$\widehat{h}(\widehat{q}_{\rm c}, r_{\rm c}) = \frac{(u_o Q - r_{\rm c}) \mathbf{1}^T W \mathbf{1}}{Q} \tag{174}$$

This shows the usual trade-off between robustness vs. critical reward, as in fig.10:

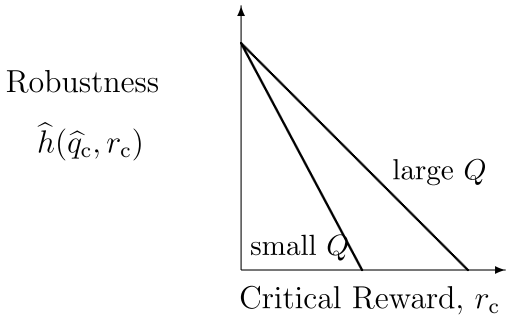

Figure 10: Robustness function vs critical reward.

Slope $\propto -\frac{1}{Q}$, where Q = total investment.
Question: Are things better or worse with large investment Q?
Answers:

- Greater robustness at fixed aspiration $r_{\rm c}$, for larger Q.
- Aspiration-cost of an increment in robustness increases as Q increases.

12.3 Comparing Portfolios

¶ Consider 2 sets of investment options, each with:
- Constant nominal return, $\tilde{u}_i = u_{o,i}\mathbf{1}$, $i = 1, 2$.
- Ellipsoid-bound info-gap models as in eq.(152) on p. 57:

$$\mathcal{U}_i(h, \tilde{u}_i) = \left\{ u = \tilde{u}_i + v : \ v^T W_i v \le h^2 \right\}, \quad h \ge 0, \ i = 1, 2 \tag{175}$$

Consider the following special case:

$$u_{o,1} < u_{o,2} \tag{176}$$

$$\mathbf{1}^T W_1 \mathbf{1} > \mathbf{1}^T W_2 \mathbf{1} \tag{177}$$

- Eq.(176) implies that option 1 is nominally worse than option 2.
- Eq.(177) implies that option 1 is nominally more certain than option 2. (Recall: W is **inverse** covariance matrix).

The optimum robustness function for investment option i is, from eq.(174) on p. 60:

$$\widehat{h}_i(\widehat{q}_{ci}, r_c) = \frac{(u_{o,i}Q - r_c)\mathbf{1}^T W_i \mathbf{1}}{Q} \tag{178}$$

¶ These two optimal robustness functions appear as in fig. 11:

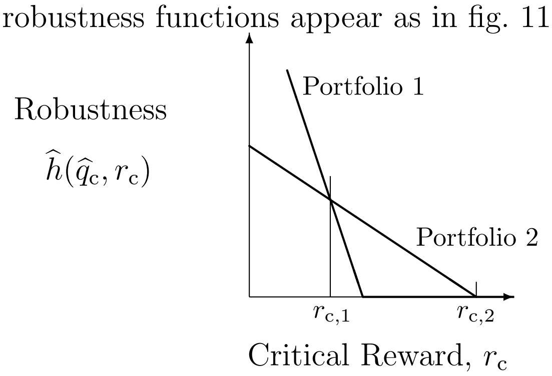

Figure 11: Robustness functions for two different portfolio investment alternatives.

Clearly:
- We prefer portfolio 1 for rewards $r_c < r_{c,1}$.
  Portfolio 2 is more risky than portfolio 1.
- We prefer portfolio 2 for rewards $r_{c,1} < r_c < r_{c,2}$.
  Portfolio 1 is more risky than portfolio 2.
- Neither portfolio is acceptable for rewards $r_{c,2} < r_c$.
  Both portfolios very risky.

12.4 Opportuneness Function

¶ We now develop the opportuneness function, $\widehat{\beta}(q, r_{\rm w})$.
$\widehat{\beta}(q, r_{\rm w}) =$ least uncertainty needed to
sustain possibility of reward
as large as $r_{\rm w}$:

$$\widehat{\beta}(q, r_{\rm w}) = \min \left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm w} \right\} \tag{179}$$

where:

$$r_{\rm w} \gg r_{\rm c} \tag{180}$$

Compare this to the robustness function, eq.(153) on p.58:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm c} \right\} \tag{181}$$

¶ $\widehat{\beta}(q, r_{\rm w})$ and $\widehat{h}(q, r_{\rm c})$ are **dual functions.**

¶ Distinct decision strategies:
$\widehat{\beta}(q, r_{\rm w})$: windfalling at $r_{\rm w}$.
$\widehat{h}(q, r_{\rm c})$: satisficing at $r_{\rm c}$.

¶ Proceeding as before we find:

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} q^T u = q^T \widetilde{u} + h \sqrt{q^T W^{-1} q} \tag{182}$$

Equate this to $r_{\rm w}$ and solve for h to find opportuneness function:

$$\widehat{\beta}(q, r_{\rm w}) = \frac{r_{\rm w} - q^T \widetilde{u}}{\sqrt{q^T W^{-1} q}} \tag{183}$$

Note trade-off of certainty vs. windfall reward.
¶ Impose the same budget constraint:

$$q^T \mathbf{1} = Q \tag{184}$$

Also, assume as before:

$$\widetilde{u} = u_o \mathbf{1} \tag{185}$$

The opportuneness function becomes:

$$\widehat{\beta}(q, r_{\rm w}) = \frac{r_{\rm w} - u_o Q}{\sqrt{q^T W^{-1} q}} \tag{186}$$

Recall the robustness function, eq.(174) on p. 60:

$$\widehat{h}(q, r_{\rm c}) = \frac{u_o Q - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{187}$$

¶ Recall "Bigger is better" for $\widehat{h}$

$\Longrightarrow$ choose q to maximize $\widehat{h}$.

"Big is bad" for $\widehat{\beta}$

$\Longrightarrow$ choose q to minimize $\widehat{\beta}$.

¶ Can we optimize $\widehat{h}$ and $\widehat{\beta}$ with the same q?

- $\max \widehat{h}$ requires minimum $q^T W^{-1} q$.
- $\min \widehat{\beta}$ requires maximum $q^T W^{-1} q$.

So we cannot simultaneously optimize $\widehat{h}$ and $\widehat{\beta}$:

Any change in q which increases $\widehat{h}$ also increases $\widehat{\beta}$.

Any change in q which decreases $\widehat{h}$ also decreases $\widehat{\beta}$.

Thus $\widehat{h}$ and $\widehat{\beta}$ are **antagonistic.**

¶ Trade-off between robustness and opportuneness.
From eqs.(186) and (187):

$$\frac{\mathrm{d}\widehat{h}(q,r_{\mathrm{c}})}{\mathrm{d}q} = -\frac{u_o Q - r_{\mathrm{c}}}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{u_o Q - r_{\mathrm{c}}}{q^T W^{-1} q}\, v \tag{188}$$

$$\frac{\mathrm{d}\widehat{\beta}(q,r_{\mathrm{w}})}{\mathrm{d}q} = -\frac{r_{\mathrm{w}} - u_o Q}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{r_{\mathrm{w}} - u_o Q}{q^T W^{-1} q}\, v \tag{189}$$

Hence:

$$\frac{\mathrm{d}\widehat{h}}{\mathrm{d}\widehat{\beta}} = \frac{u_o Q - r_{\mathrm{c}}}{r_{\mathrm{w}} - u_o Q} > 0 \tag{190}$$

The trade-off between robustness and opportuneness is shown schematically in fig. 12.

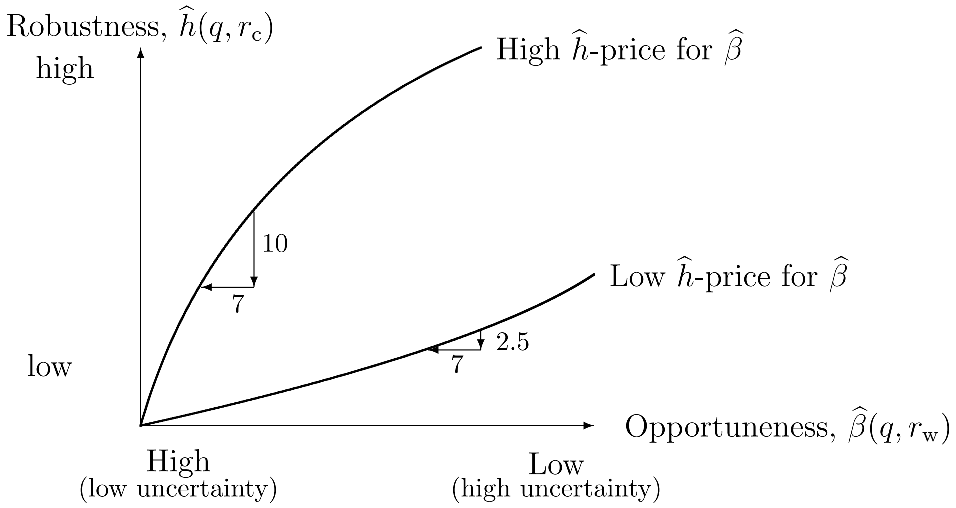

Figure 12: Trade-off between robustness and opportuneness.

¶ Does $\widehat{\beta}$ have an optimum?
Can we maximize $q^T W^{-1} q$ subject to $q^T \mathbf{1} = Q$?
No. See fig. 13.
For any constant $= q^T W^{-1} q$
There is a q which also satisfies the constraint.
However, as q moves far from the origin,
other constraints become active.

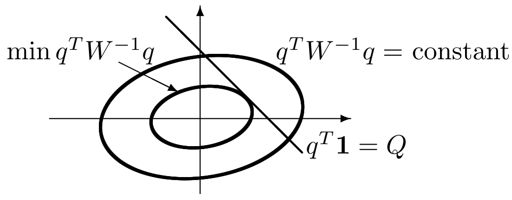

Figure 13: Schematic illustration of constrained optimization of $q^T W^{-1} q$.

13 Search and Evasion

¶ Tracking problem:
- Intelligent "hunter" tries to catch an intelligent "prey".
- Examples:
  - Homing missile.
  - Robotic grasping.
  - Job search.

¶ Dynamics:
- Hunter and prey move on a line.
- $x(t)$ = hunter's position. $x(0) = 0$.
- $u(t)$ = prey's position. $u(0) > 0$.
- The hunter measures prey's position but hunter does not know prey's evasion strategy.
- Hunter moves according to:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = q\,[u(t) - x(t)] \tag{191}$$

q = constant which hunter chooses before chase.

¶ Hunter has limited info about prey's evasive strategy:
• $\widetilde{s}$ = typical speed.
• Actual speed differs from $\widetilde{s}$ by unknown constant.
• Hunter's slope-bound info-gap model:

$$\mathcal{U}(h,\widetilde{s}) = \left\{ u(t) : \ \left| \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \widetilde{s} \right| \le h \right\}, \quad h \ge 0 \tag{192}$$

¶ Performance requirement:
The hunter is successful if, at a specified time T,
the hunter-prey distance $\le \Delta$:

$$|x(T) - u(T)| \le \Delta \tag{193}$$

¶ Hunter must choose q in eq.(191) to:
• maximize robustness to uncertain prey behavior.
• satisfying performance requirement in (193).

¶ Robustness function $\widehat{h}(q,\Delta)$:

$$\widehat{h}(q,\Delta) = \max \left\{ h : \ \max_{u \in \mathcal{U}(h,\widetilde{s})} |x(T) - u(T)| \le \Delta \right\} \tag{194}$$

¶ Dynamics again: solution of eq.(191) is:

$$x_u(t) = q \int_0^t \mathrm{e}^{-q(t-\tau)} u(\tau)\, \mathrm{d}\tau \tag{195}$$

After manipulation, including a partial integration, eq.(195) is:

$$\begin{aligned} x_u(t) - u(t) \ &= \ -\mathrm{e}^{-qt} u(t) - \mathrm{e}^{-qt} \int_0^t (\mathrm{e}^{q\tau} - 1) \left(\frac{\mathrm{d}u}{\mathrm{d}\tau} - \widetilde{s} \right) \mathrm{d}\tau \\ & \quad -\frac{\widetilde{s}}{q} \left(1 - \mathrm{e}^{-qt} \right) + \widetilde{s} t \mathrm{e}^{-qt} \end{aligned} \tag{196}$$

¶ Max absolute distance if prey adopts: $u(t) = u(0) + (\widetilde{s} + h)t$:

$$\max_{u \in \mathcal{U}(h,\widetilde{s})} |x_u(t) - u(t)| = \mathrm{e}^{-qt} u(0) + \frac{\widetilde{s} + h}{q} \left(1 - \mathrm{e}^{-qt} \right) \tag{197}$$

¶ Robustness: equate eq.(197) to Δ and solve for h:

$$\widehat{h}(q,\Delta) = \frac{\left(\Delta - \mathrm{e}^{-qT} u(0) \right) q}{1 - \mathrm{e}^{-qT}} - \widetilde{s} \tag{198}$$

unless this is negative, in which case the robustness is zero.

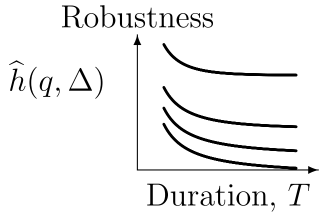

Figure 14: Robustness versus time, eq.(198), assuming $u(0) < \Delta$. The value of q increases from the bottom to the top curve.

¶ Results: eq.(198) is plotted in fig. 14.

- q increases from the bottom to the top curve.
- q is a measure of hunter's effort:
  - Large q implies large effort.
  - Large q implies large robustness.
- $\widehat{h}(q,\Delta)$ decreases with chase time T if $u(0) < \Delta$:
  Long chase has low robustness.
- Choose q according to:
  - required robustness.
  - required chase duration.

¶ Return to eq.(198) on p. 67. We see that:

$$\frac{\partial \widehat{h}}{\partial T} > 0 \ \text{if} \ u(0) > \Delta \tag{199}$$

$$\frac{\partial \widehat{h}}{\partial T} < 0 \ \text{if} \ u(0) < \Delta \tag{200}$$

Meaning:
- Robustness increases in time, eq.(199), when chasing "distant" prey.
- Robustness decreases in time, eq.(200), in ambush.

14 Assay Design: Environmental Monitoring

14.1 Measuring Biomass

§ This section is based on section 3.2.10 in:
Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

§ **The problem:**
• The local municipality will release waste into the river.
• We must design a monitoring system to detect contamination.
• The monitoring system measures local biomass at each of N locations along the river.
• We *wish* to trigger an alarm if the total biomass downstream of the release exceeds B_c.
• We *will actually* trigger an alarm if the local biomass exceeds a critical value, ρ_0, at one or more measurement sites.
• The biomass density distribution, $\rho(x)$, is highly uncertain.
• *Design task:* choose N and ρ_0.
• Method: evaluate robustness to spatial uncertainty in $\rho(x)$.

§ **Information about the spatial uncertainty.**
• $\rho(x)$, varies gradually along the length of the river.
• Maximum slope of $\rho(x)$ no more extreme than $\pm\widetilde{s}$.
• $\widetilde{s}$ is highly uncertain.

§ **The slope-bound info-gap model.**
• Include the no-alarm assay result that density is no greater than ρ_0 at all of the N test points x_i :

$$\mathcal{U}(h, \rho_0, \widetilde{s}) = \left\{ \rho(x) : \ \rho(x_i) \le \rho_0, i = 1, \ldots, N; \ \left| \frac{|\rho'(x)| - \widetilde{s}}{\widetilde{s}} \right| \le h \right\}, \ h \ge 0 \tag{201}$$

The inequality on ρ' means that, at horizon of uncertainty h, $\rho'(x)$ satisfies one of:

$$\text{positive slope:} \quad (1-h)\widetilde{s} \le \quad \rho'(x) \quad \le (1+h)\widetilde{s} \tag{202}$$

$$\text{negative slope:} \quad -(1+h)\widetilde{s} \le \quad \rho'(x) \quad \le (-1+h)\widetilde{s} \tag{203}$$

However, the horizon of uncertainty, h, is unknown.
• Note: the info-gap model depends on the design (N, ρ_0) and on the fact that the observations ($\rho(x_i)$, $i = 1, \ldots, N$) are all “okay”.

§ **Robustness** of N measurement sites, trigger level ρ_0, with critical total mass $B_{\rm c}$:

$$\widehat{h}(N, \rho_0, B_{\rm c}) = \max\left\{ h : \underbrace{\left(\max_{\rho \in \mathcal{U}(h, \rho_0, \widetilde{s})} \int_0^L \rho(x)\, {\rm d}x \right)}_{M(h)} \le B_{\rm c} \right\} \tag{204}$$

§ **Evaluating the robustness: conceptual.**

- $M(h)$ is defined in eq.(204).
- $M(h)$ increases monotonically as h increases.
- Hence $M(h)$ is the inverse of $\widehat{h}(N, \rho_0, B_{\rm c})$:

$$M(h) = B_{\rm c} \quad \text{implies} \quad \widehat{h}(N, \rho_0, B_{\rm c}) = h \tag{205}$$

- A plot of h (vertical) vs. $M(h)$ (horizontal)
  is the same as a plot of
  $\widehat{h}(N, \rho_0, B_{\rm c})$ (vertical) vs. $B_{\rm c}$ (horizontal).

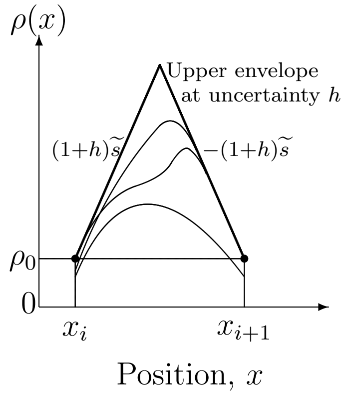

Figure 15: Evalution of $M(h)$, eq.(204), showing an upper envelope and three possible density curves.

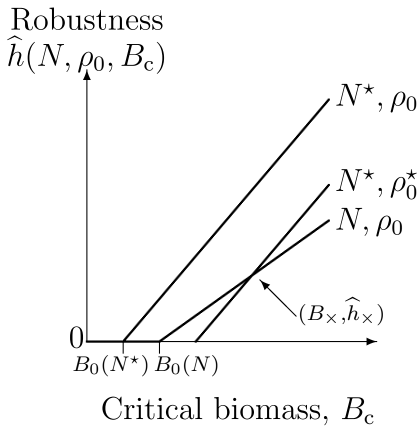

Figure 16: Robustness curves for N and $N^\star$ test points with trigger densities ρ_0 and $\rho_0^\star$, eq.(207). $N^\star > N$, $\rho_0^\star > \rho_0$.

§ **Evaluating the robustness,** (fig. 15):

• Given measured densities of ρ_0 at adjacent test points.

• Max biomass occurs at extremal slopes of $\rho(x)$.

• Max biomass at horizon of uncertainty h, in the $N-1$ equal intervals between 0 and L, is:

$$M(h) = L\rho_0 + \frac{L^2 \widetilde{s}}{4(N-1)}(1+h) \tag{206}$$

Equate eq.(206) to the critical biomass B_c and solve for h yields robustness:

$$\widehat{h}(N,\rho_0,B_c) = \begin{cases} \dfrac{4(N-1)}{L^2\widetilde{s}}(B_c - L\rho_0) - 1 & \text{if } B_c \geq \underbrace{L\rho_0 + \dfrac{L^2\widetilde{s}}{4(N-1)}}_{B_0(N)} \\ 0 & \text{else} \end{cases} \tag{207}$$

§ **Trade-offs**:

• Robustness increases ($\widehat{h}$ gets larger) as the performance gets worse (B_c gets larger), fig. 16.

• Robustness increases with increase in the number of test points in the length L along the river.

• Robustness increases as the alarm threshold, ρ_0, gets smaller.

§ **Unreliability of estimated performance,** fig. 16.

- $B_0(N)$ in eq.(207) is the biomass of a distribution whose:
  - measurements all equal ρ_0 and,
  - slope between test points equals the anticipated values of $\pm\widetilde{s}$.
- This nominal biomass has zero robustness of detection:

$$\widehat{h}(N, \rho_0, B_c) = 0 \quad \text{if} \quad B_c = B_0(N) \tag{208}$$

§ **Preference reversal.**

- Note crossing robustness curves in fig. 16 for $N < N^\star$ and $\rho_0 < \rho_0^\star$.
- That is, reducing # of measurements can be compensated for
  by reducing the trigger density,
  at constant robustness to spatial uncertainty.

§ **Demanded robustness.**

- $\widehat{h}_d$ denotes demanded robustness to slope-uncertainty.
- E.g., $\widehat{h}_d = 0.5$ implies:
  Estimated max slope, $\widetilde{s}$, can err up to 50%
  without jeopardizing missed detection of excess biomass.
- Choose N and ρ_0 to satisfy:

$$\widehat{h}(N, \rho_0, B_c) = \widehat{h}_d \tag{209}$$

14.2 Choosing Sample Size: Special Case of Small Effect Size

§ This section is a special case of a more general problem studied in:
David R. Fox, Yakov Ben-Haim, Keith R. Hayes, Michael McCarthy, Brendan Wintle, Piers Dunstan, 2007, An info-gap approach to power and sample size calculations, *Environmetrics*, vol. 18, pp.189–203.

§ **Notation:**
- x = a statistic, e.g. sample mean.
- $f(x)$ = sampling distribution of x. Uncertain.
- $\widetilde{f}(x)$ = Best-estimate of the sampling distribution of x.
- δ = effect size: suspected change in the value estimated by x.

§ **Binary decision:**
- Null hypothesis: there was no change .
- Alternative hypothesis: there was a change greater than δ.
- Threshold test with "critical value" C: Decide "no change" iff $x \leq C$.
- α = Level of significance,
  = probability of falsely rejecting the null hypothesis.

$$1 - \alpha \ = \ \int_{-\infty}^{C} f(x)\,\mathrm{d}x \tag{210}$$

- $\beta(f)$ = 1 minus the power,
  = probability of falsely rejecting the alternative hypothesis.

$$\beta(f) \ = \ \int_{-\infty}^{C} f(x-\delta)\,\mathrm{d}x = \int_{-\infty}^{C-\delta} f(x)\,\mathrm{d}x = 1 - \alpha - \int_{C-\delta}^{C} f(x)\,\mathrm{d}x \tag{211}$$

§ **Power of the test:**
- Power = $1 - \beta$:

$$1 - \beta = \int_{C}^{\infty} f(x-\delta)\,\mathrm{d}x \tag{212}$$

- Power is probability of *correctly* rejecting H_0.
- Compare with Level of significance: probability of *falsely* rejecting H_0.
- We want both α and β to be *small*.

§ **Standard statistical approach** to determining the sample size:
- Know sampling distribution, $f(x)$.
- $f(x)$ depends on the number of measurements.
- Specify level of significance α and the effect size δ.
- Evaluate the critical value and the power from eqs.(210) and (211).
- Increase the number of measurements until the power is adequate.

§ **The problem:** $f(x)$ is highly uncertain.

§ **Fractional-error info-gap model:**

$$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) : \ f \in \mathcal{P}, \ |f(x) - \widetilde{f}(x)| \leq h\widetilde{f}(x) \right\}, \quad h \geq 0 \tag{213}$$

$\mathcal{P}$ is the set of all non-negative and normalized pdfs on the domain of x.

§ **Critical value:**

- Choose critical value based on estimated distribution:
- Let $\widetilde{C}$ denote the $1-\alpha$ quantile of the nominal distribution $\widetilde{f}(x)$.

§ **Analyst's requirement.**

- β needs to be small.
- Let $1-\beta_{\rm d}$ be the power which is demanded by the analyst. That is, the analyst requires $\beta \le \beta_{\rm d}$.

§ **The decision:**

- Choose sample size, N.
- Strategy: **robust-satisficing**:
  - Satisfice the power.
  - Maximize the robustness.

§ **The robustness** of N measurements, with the requirement $\beta_{\rm d}$, is:

$$\widehat{h}(N, \beta_{\rm d}) = \max \left\{ h : \ \left(\max_{f \in \mathcal{U}(h, \widetilde{f})} \beta(f) \right) \le \beta_{\rm d} \right\} \tag{214}$$

§ **Small Effect Size:**

$$\delta \ll 1 \tag{215}$$

Now eq.(211) can be approximated as:

$$\beta(f) = 1 - \alpha - f(C)\delta \tag{216}$$

§ **Inner max in eq.(214).**

- The pdf in $\mathcal{U}(h, \widetilde{f})$ which maximizes β is very nearly:

$$\widehat{f}(x) = \begin{cases} \widetilde{f}(x) & \text{if } x < \widetilde{C} - \delta \\ (1-h)\widetilde{f}(x) & \text{if } x \in [\widetilde{C} - \delta, \ \widetilde{C}] \\ (1+wh)\widetilde{f}(x) & \text{if } x > \widetilde{C} \end{cases} \tag{217}$$

where w is a very small positive number which normalizes $\widehat{f}(x)$. That is, w is determined so that the decrement in $\widehat{f}$ in $[\widetilde{C} - \delta, \ \widetilde{C}]$ is compensated by the increment in $(\widetilde{C}, \infty)$:

$$wh[1 - \widetilde{F}(\widetilde{C})] = h\delta \widetilde{f}(\widetilde{C}) \tag{218}$$

where $\widetilde{F}$ is the cumulative distribution function of $\widetilde{f}$.

- The inner max in eq.(214) is $\beta(\widehat{f})$ from eq.(216) and (217):

$$\beta(\widehat{f}) = 1 - \alpha - (1-h)\widetilde{f}(\widetilde{C})\delta \tag{219}$$

which is the greatest value of β at horizon of uncertainty h.

§ **Robustness.**
Equate eq.(219) to the demanded value, $\beta_{\rm d}$, and solve for h for robustness of N measurements:

$$\widehat{h}(N,\beta_{\rm d}) = \left\{ \begin{array}{cl} 0 & \text{if } \beta_{\rm d} < 1-\alpha-\widetilde{f}(\widetilde{C})\delta \\ \dfrac{\beta_{\rm d}-1+\alpha+\widetilde{f}(\widetilde{C})\delta}{\widetilde{f}(\widetilde{C})\delta} & \text{else} \end{array} \right. \tag{220}$$

§ **Discussion:**
• The robustness increases as $\beta_{\rm d}$ increases.
• The robustness is zero when $\beta_{\rm d}$ equals the nominal value, $\beta(\widetilde{f})$.
• This derivation is contingent on the small-effect assumption in eq.(215).
• The dependence of the robustness on the sample size arises through the nominal sampling distribution at the $1-\alpha$ quantile, $\widetilde{f}(\widetilde{C})$.

15 Strategic Asset Allocation

§ This section based on section 4.4 of Yakov Ben-Haim, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave.

§ **Generic idea of an asset:**
- Energy supply to different actuators: motion on complex terrain; robotics.
- Load points for deflection, especially in non-linear system.
- Search locations (looking for treasure or enemies).
- Innovative ideas or projects.
- Stocks or bonds in finance: monetary return.

§ **Generic idea of strategic allocation:**
- Dynamic setting: multiple time steps.
- Allocation at each time step.
- "Returns" or "outcomes" at each step determine resources for next step.

§ **Basic idea of asset allocation ("financial" model):**
- Choose an allocation of resources (e.g. budget) between different assets.
- The future returns are random and the pdf is uncertain.
- You require high probability that the future balance is acceptable.
  That is, the future **capital reserve** (or profit) must be adequate with high probability.

15.1 Budget Constraint

Basic variables:

x_{it} is the **quantity of the ith asset which is purchased** at time t. x_{it} can be either positive or negative. The allocation vector is $x_t = (x_{1t}, \ldots, x_{Nt})^T$. This is **chosen at time t.**

p_{it} is the **ex-dividend price of the ith asset** for purchase at time t. The vector of prices is $p_t = (p_{1t}, \ldots, p_{Nt})^T$. **Known at time t.**

y_{it} is the **payoff of the ith asset** at time $t+1$. The vector of payoffs is $y_t = (y_{1t}, \ldots, y_{Nt})^T$. **Not known at time t.**

c_t is the **capital reserve** of the financial institution[2] at time $t+1$. **Not known at time t.**

The **budget constraint:**

$$c_t + p_t^T x_t = y_t^T x_{t-1} \tag{221}$$

[2]For an individual investor c_t could be thought of as consumption.

15.2 Uncertainty

§ **Moderate uncertainty:**

- y_t, is random and known to be normally distributed.
- Moments are estimated but uncertain:
  - Estimated mean of the payoff vector is μ_{yt}.
  - Estimated covariance matrix of the payoff is Σ_{yt}.

§ Thus, from the budget constraint in eq.(221), the capital reserve is a normal random variable with estimated mean and variance:

$$\widetilde{\mu}_{ct} = -p_t^T x_t + \mu_{yt}^T x_{t-1} \tag{222}$$

$$\widetilde{\sigma}_{ct}^2 = x_{t-1}^T \Sigma_{yt} x_{t-1} \tag{223}$$

§ Error values of the estimated mean and standard deviation, $\widetilde{\mu}_{ct}$ and $\widetilde{\sigma}_{ct}$, are ε_μ and ε_σ.

§ Info-gap model for uncertainty in the distribution of the capital reserve, c_t:

$$\mathcal{U}(h) = \left\{ f(c_t) \sim N(\mu_{ct}, \sigma_{ct}^2) : \ \left| \frac{\mu_{ct} - \widetilde{\mu}_{ct}}{\varepsilon_\mu} \right| \le h, \ \left| \frac{\sigma_{ct} - \widetilde{\sigma}_{ct}}{\varepsilon_\sigma} \right| \le h, \ \sigma_{ct} \ge 0 \right\}, \quad h \ge 0 \tag{224}$$

15.3 Performance and Robustness

Performance requirement.
The α **quantile** of the distribution $f(c_t)$, denoted $q(\alpha, f)$, is the value of c_t for which the probability of being less than this value equals α. This quantile is defined in:

$$\alpha = \int_{-\infty}^{q(\alpha,f)} f(c_t)\,\mathrm{d}c_t \tag{225}$$

α is typically small so $q(\alpha, f)$ may be negative.

§ The **performance requirement** is:

$$q(\alpha, f) \geq r_{\mathrm{c}} \tag{226}$$

We will use the robustness function to evaluate the confidence in satisfying this requirement for chosen investment, x_t.

Robustness function:

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \max\left\{h: \ \left(\min_{f\in\mathcal{U}(h)} q(\alpha, f)\right) \geq r_{\mathrm{c}}\right\} \tag{227}$$

§ z_α is the α quantile of the standard normal distribution.
- Assume: $\alpha < 1/2$ so that $z_\alpha < 0$.
- Typically α around 0.01.

§ One can show:

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \frac{r_{\mathrm{c}} - q(\alpha, \widetilde{f})}{\varepsilon_\sigma z_\alpha - \varepsilon_\mu} \tag{228}$$

or zero if this is negative.
- The numerator and denominator are both negative, so the robustness decreases as r_{c} increases towards $q(\alpha, \widetilde{f})$.

15.4 Opportuneness Function

§ **Windfall aspiration is:**

$$q(\alpha, f) \geq r_{\rm w} > r_{\rm c} \tag{229}$$

§ **Opportuneness:**

$$\widehat{\beta}(x_t, r_{\rm w}) = \min \left\{ h : \ \left(\max_{f \in \mathcal{U}(h)} q(\alpha, f) \right) \geq r_{\rm w} \right\} \tag{230}$$

§ Inverse of opportuneness:

- $M(h)$ denotes the **inner maximum** in eq.(230).
- $M(h)$ is the **inverse of the opportuneness.**
- That is, a plot of $M(h)$ vs. h is the same as a plot of $r_{\rm w}$ vs. $\widehat{\beta}(x_t, r_{\rm w})$.
- We will derive an explicit expression from which to evaluate $M(h)$.

§ **Ramp function:** $r(x) = 0$ if $x < 0$ and $r(x) = x$ if $x \geq 0$.

§ **One assumption:**

- z_α is the α quantile of the standard normal distribution.
- We assume that $\alpha < 1/2$, so that $z_\alpha < 0$.

§ **One can show:**

$$q(\alpha, f) = \sigma_{ct} z_\alpha + \mu_{ct} \tag{231}$$

Proof:

$$\alpha = {\rm Prob}\,(x \leq q(\alpha, f)) \tag{232}$$

$$= {\rm Prob}\left(\frac{x - \mu_{ct}}{\sigma_{ct}} \leq \frac{q(\alpha, f) - \mu_{ct}}{\sigma_{ct}} \right) \tag{233}$$

Note that:

$$z = \frac{x - \mu_{ct}}{\sigma_{ct}} \sim \mathcal{N}(\mu_{ct}, \sigma_{ct}) \tag{234}$$

$$z_\alpha = \frac{q(\alpha, f) - \mu_{ct}}{\sigma_{ct}} \tag{235}$$

Re-arranging eq.(235) leads to eq.(231).

§ **Inverse of opportuneness function:**

$$M(h) = r(\widetilde{\sigma}_{ct} - \varepsilon_\sigma h) z_\alpha + \widetilde{\mu}_{ct} + \varepsilon_\mu h \tag{236}$$

15.5 Policy Exploration

§ **Example:**

• One risk-free asset, $i = 1$, and a one uncorrelated risky asset, $i = 2$.
• Select the allocation.
• Price vector is $p_t = (7, 10)$.
• The level of confidence of the quantile is $\alpha = 0.01$.
• The standard deviation of the payoff of the risky asset is 5% of its estimated mean unless indicated otherwise.
• Thus $(\Sigma_{yt})_{22} = (0.05\mu_{yt,2})^2$. The other elements of the 2×2 covariance matrix Σ_{yt} are zero.

§ **Trade-offs and zeroing** (fig. 17):

• Robustness vs critical reserve.
• Opportuneness vs windfall reserve.

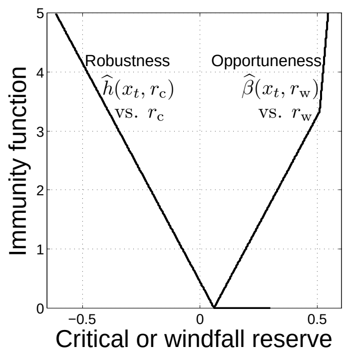

Figure 17: Robustness and opportuneness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. $\mu_{yt} = (1.04p_{1t}, 1.08p_{2t})^T$. $\varepsilon_\mu = 0.05\widetilde{\mu}_{ct}$. $\varepsilon_\sigma = 0.3\widetilde{\mu}_{ct}$.

Port-folio	$\mu_{yt,1}/p_{1t}$	$\mu_{yt,2}/p_{2t}$	$\widetilde{\mu}_{ct}$	$\widetilde{\sigma}_{ct}$	$\varepsilon_\mu/\widetilde{\mu}_{ct}$	$\varepsilon_\sigma/\widetilde{\sigma}_{ct}$
1	0.04	0.08	0.436	0.162	0.05	0.1
2	0.036	0.076	0.404	0.161	0.035	0.075

Table 1: Parameters of two portfolios. Robustness curves in fig. 18.

Choose between two portfolios, table 1.

- First portfolio has higher estimated mean payoffs and higher errors.
- Classical dilemma: portfolio 1 is better on average, but more uncertain.

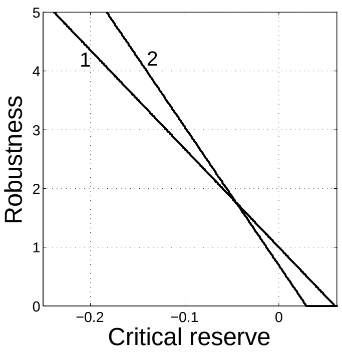

Figure 18: Robustness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. See table 1.

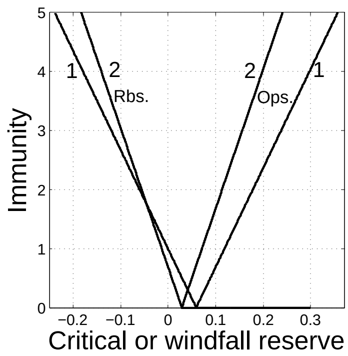

Figure 19: Robustness and opportuneness curves for portfolios in fig. 18.

§ **Preference reversal,** fig. 18.

§ **Robustness and opportuneness,** fig. 19.

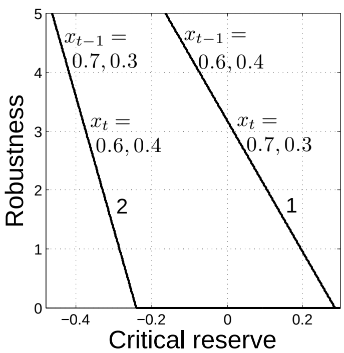

Figure 20: Robustness curves for two sequences of investments.

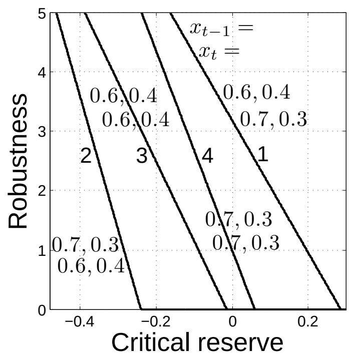

Figure 21: Robustness curves for 4 sequences of investments. Curves 1 and 2 reproduced from fig. 20.

§ **Sequence matters,** fig. 20.

- Sequence of investment vectors are reversed between the two portfolios.
- Two differences between outcomes:
  - Portfolio 1 has much higher nominal α quantile (horizontal intercept).
  - Portfolio 2 has steeper slope, which implies lower cost of robustness.

§ **Sequence matters,** fig. 21.

- Portfolios 1 and 2 same as fig. 20.
- Portfolio 3 and 4 are similar, and without investment change over time.