Problem Set on

Info-Gap Risks in Project Management

List of problems:

1. **Planning for uncertain costs.** Consider a project with N tasks whose costs are $c = (c_1, \ldots, c_N)^T$ and which are initiated at times $t = (t_1, \ldots, t_N)^T$. The total project duration is T. The money needed for each task must be available when that task is initiated, and will be funded by a loan at interest rate r. The loan will be repaid at the end of the project. The total project cost, including interest, is:

$$C(c,t) = \sum_{i=1}^{N} c_i \mathrm{e}^{r(T-t_i)} \tag{1}$$

We have very limited information about the task costs. We have estimates from the past of the mean costs and the variances and co-variances of the costs. The historical mean vector and covariance matrix are $\tilde{c}$ and V. We will consider two different info-gap models for uncertainty in the task costs.

1st info-gap model. We will use an ellipsoid-bound info-gap model to represent uncertain future realizations of the cost vector:

$$\mathcal{U}(\alpha, \tilde{c}) = \left\{ c : \ (c - \tilde{c})^T V^{-1} (c - \tilde{c}) \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{2}$$

2nd info-gap model. We assume that the covariances are all zero (or we simply ignore covariance information), so V is a diagonal matrix of variances v_i. The fractional-error info-gap model is:

$$\mathcal{U}(\alpha, \tilde{c}) = \left\{ c : \ |c_i - \tilde{c}_i| \le \alpha \sqrt{v_i}, \ i = 1, \ldots, N \right\}, \quad \alpha \ge 0 \tag{3}$$

(a) Use the info-gap model of eq.(2) to derive an expression for the robustness, to uncertainty in task costs, of task-initiation times t. The manager aspires to keep the total project cost below the critical value C_{f}.

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(b) Now suppose that the covariance matrix V is diagonal with elements $v_1, \ldots, v_N$. Use the results of question 1a to indicate how one might wish to choose the task-initiation times t.

(c) Repeat question 1a with the info-gap model of eq.(3).

(d) Repeat question 1b with the info-gap model of eq.(3).

2. **Adaptive planning for uncertain costs.** In problem 1 we considered planning task-initiation times given task-cost uncertainty and budget constraint. We included the effect of time by considering the net present value (at the time of planning) of future expenditures. Now modify that problem to consider the adaptive determination, "on-line" or in "real time", of the task-initiation times. That is, at any moment of time *during* implementation of the project, a certain number of tasks have already been funded, a certain amount of the total budget has already been spent, a certain fraction of the total project time T remains, and the remaining tasks must be implemented with the remaining budget. Using the info-gap models of eqs.(2) or (3), formulate a robustness function for the adaptive determination of task-initiation time.

3. **Budget allocation.** Consider a project with N tasks. After budget allocation for basic operational needs of the N tasks, a quantity Q of money remains for allocation among the tasks, on the basis of their anticipated contribution to project profitability. The anticipated rate of return of each additional dollar allocated to task i is $\widetilde{u}_i$, for $i = 1, \ldots, N$. Thus an allocation q is anticipated to result in added revenue:

$$R(q, \widetilde{u}) = \sum_{i=1}^{N} q_i \widetilde{u}_i \tag{4}$$

The allocation q has succeeded if the added revenue is no less than R_c. However, the anticipated return vector $\widetilde{u}$ is highly uncertain. The actual rates of return, u_i, are unknown, and the uncertainty in the vector of rates of return u is denoted by an info-gap model, $\mathcal{U}(\alpha, \widetilde{u})$. Consider two info-gap models:

$$\mathcal{U}(\alpha, \widetilde{u}) = \{u : \ |u_i - \widetilde{u}_i| \le \alpha w_i, \ i = 1, \ldots, N\}, \quad \alpha \ge 0 \tag{5}$$

where the uncertainty weights w_i are known positive numbers, and:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{u : \ (u - \widetilde{u})^T V^{-1} (u - \widetilde{u}) \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{6}$$

where V^{-1} is a known, positive definite, real, symmetric matrix.

Assume that no allocations are negative, so $q_i \ge 0$ for all i.

(a) Use the info-gap model of eq.(5) to suggest how one might go about choosing an allocation of Q among the tasks, based on the robustness function.

(b) Repeat question 3a based on the opportuneness function.

(c) Repeat question 3a based on the info-gap model of eq.(6).

(d) ‡ Now suppose that we have an estimated joint probability density (pdf) for the rates of return, $\widetilde{p}(u)$. Specifically, suppose that the estimated pdf of the revenue from the ith task is normal with mean μ_i and variance σ_i^2, and that the revenues of different tasks are uncorrelated statistically:

$$\widetilde{p}_i(u_i) \sim \mathcal{N}(\widetilde{\mu}_i, \widetilde{\sigma}_i^2), \quad i = 1, \ldots, N \tag{7}$$

$$\widetilde{p}(u) = \prod_{i=1}^{N} \widetilde{p}_i(u_i) \tag{8}$$

The moments $\widetilde{\mu}_i$ and $\widetilde{\sigma}_i^2$ are highly uncertain, and the uncertainty is represented by an info-gap model:

$$\mathcal{U}(\alpha, \widetilde{p}_i) = \left\{ \mu_i, \sigma_i : \left| \frac{\mu_i - \widetilde{\mu}_i}{\widetilde{\mu}_i} \right| \le \alpha, \ \max[0, \ (1-\alpha)\widetilde{\sigma}_i] \le \sigma_i \le (1+\alpha)\widetilde{\sigma}_i \right\}, \quad \alpha \ge 0 \quad (9)$$

For any pdf $p(u)$, the probability of success of allocation q is:

$$P_{\mathrm{s}}(q, p) = \mathrm{Prob}[R(q, u) \ge R_{\mathrm{c}}] \quad (10)$$

The project owners require that the probability of success be no less than P_{c}. Formulate an expression for the robustness, to uncertainty in the pdf, of the probability of success of allocation q. Develop this expression as far as you can, and use it to choose the allocation q.

4. **Project termination.** At any point in time, the project manager knows the total cumulative expenditure for the project, E. In addition, the manager knows what fraction f_i of task i remains to be completed, for each task $i = 1, \ldots, N$. Denote $f = (f_1, \ldots, f_N)^T$.

The estimated time remaining before completion of the project is:

$$t = \widetilde{g}^T f \quad (11)$$

where $\widetilde{g}$ is highly uncertain and its error is represented by an info-gap model of uncertainty:

$$\mathcal{U}_g(\alpha, \widetilde{g}) = \left\{ g : \ (g - \widetilde{g})^T W (g - \widetilde{g}) \le \alpha^2 \right\}, \quad \alpha \ge 0 \quad (12)$$

W is a known, real, symmetric, positive definite matrix.

Given an estimate of the time remaining before completion of the project, the projected remaining cost of the project is:

$$\widetilde{c}(t) = c_0 \sqrt{t} \quad (13)$$

This projected remaining cost is highly uncertain and its error is estimated by an info-gap model of uncertainty:

$$\mathcal{U}_c(\alpha, \widetilde{c}) = \{ c(t) : \ |c(t) - \widetilde{c}(t)| \le \alpha \widetilde{c}(t) \}, \quad \alpha \ge 0 \quad (14)$$

The project fails if the total expenditure at the end of the project exceeds the budget B, which is specified.

(a) Given the manager's knowledge of E and f at a particular point in time during the implementation of the project, how confident is the manager that the project will remain within the budget? Should the manager recommend that the project be terminated? How much additional budget would be needed to make in-budget completion fairly certain? How much budget could be removed from the project without jeopardizing successful in-budget completion?

(b) The manager is in fact responsible for two projects, the project described above and an additional project with similar structure. For project $k = 1$ or 2, we denote the quantities E_k, f^k, g^k, $\widetilde{g}^k$, W_k, $c_{0,k}$ and B_k. Each project has info-gap models with the same structure. At a particular point in time during the implementation of both projects, the manager is informed of the status of the two projects by learning the values E_k and f^k for $k = 1$ and 2. The manager can move funds between the projects. How much money should be moved between the projects, if any? Should the manager recommend termination of one project and transfer of all its funds to the other project?

(c) Repeat part 4a with the following modification. Given an estimate of the time remaining before completion of the project, based on eq.(11), the remaining cost $c(t)$ is a random variable. The estimate of the pdf of $c(t)$ is:

$$\widetilde{p}(c) = \frac{1}{\widetilde{c}} \mathrm{e}^{-c/\widetilde{c}}, \quad c \geq 0 \tag{15}$$

where $\widetilde{c}$ is given in eq.(13). This pdf is highly uncertain and its error is represented by an info-gap model:

$$\mathcal{U}_p(\alpha, \widetilde{p}) = \{p(c) : \ p(c) \in \mathcal{P}, \ |p(c) - \widetilde{p}(c)| \leq \alpha \widetilde{p}(c)\}, \quad \alpha \geq 0 \tag{16}$$

where $\mathcal{P}$ is the set of non-negative pdfs normalized on $[0, \infty)$.

5. **Budgeting the manager's time.** The project manager must allocate his or her time among N tasks. The planned allocation is $\widetilde{t} = (\widetilde{t}_1, \ldots, \widetilde{t}_N)^T$. In practice, the actual time which the manager devotes to the N tasks will be $t = (t_1, \ldots, t_N)^T$. The uncertainty in the manager's attention times is represented by the following info-gap model:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \ (t - \widetilde{t})^T V (t - \widetilde{t}) \leq \alpha^2 \right\}, \quad \alpha \geq 0 \tag{17}$$

where V is real, symmetric, positive definite and known. The utility from time-allocation t is $y^T t$ where y is a known vector. It is required that the utility be no less than u_{c}. Develop an explicit analytical expression for the robustness to uncertainty in $\widetilde{t}$.

6. **Uncertain budget.** The budget for each of the N tasks in a project is $b = (b_1, \ldots, b_N)^T$, where each element of this vector can be any real number. b is unknown and its uncertainty is represented by:

$$\mathcal{U}(\alpha) = \left\{ b : \ b^T b \leq \alpha^2 \right\}, \quad \alpha \geq 0 \tag{18}$$

The overall dis-utility to the firm of budget b is expressed by $b^T F b$ where F is a real, symmetric and positive definite matrix. It is required that the overall dis-utility be no greater than u_{c}. The firm's management will determine the matrix F. Derive an explicit algebraic expression for the robustness to uncertainty in the budget, given the matrix F.

7. **Hybrid uncertainty: uncertain probabilistic profit.** The profit from a new project is a random variable x which is distributed exponentially:

$$P(x|\lambda) = 1 - \mathrm{e}^{-\lambda x}, \quad x \geq 0 \tag{19}$$

The estimated value of λ is $\widetilde{\lambda}$, but this estimate is highly uncertain:

$$\mathcal{U}(\alpha, \widetilde{\lambda}) = \left\{ \lambda : \ \lambda > 0, \ |\lambda - \widetilde{\lambda}| \leq \alpha \widetilde{\lambda} \right\}, \quad \alpha \geq 0 \tag{20}$$

It is required that profit at least x_{c} occur with probability no less than P_{c}:

$$\mathrm{Prob}(x \geq x_{\mathrm{c}} | \lambda) \geq P_{\mathrm{c}} \tag{21}$$

Derive an explicit algebraic expression for the robustness to uncertainty in λ.

8. **Project profitability.** The profit from a project is described by:

$$R(q, u) = q_1 u + q_2 \tag{22}$$

where $q = (q_1, q_2)$ are positive numbers chosen by the manager. The parameter u is uncertain:

$$\mathcal{U}(\alpha, \widetilde{u}) = \{u : \ |u - \widetilde{u}| \leq \alpha\}, \quad \alpha \geq 0 \tag{23}$$

The profit must be at least R_{c}. Profit as large as R_{w} would be wonderful. Derive expressions for the robustness and opportuneness functions.

9. **Customer satisfaction.** The satisfaction of the customer is measured as:

$$S = qA + q^2 B \tag{24}$$

where q is controlled by the manager and A and B are uncertain:

$$\mathcal{U}(\alpha, \widetilde{A}, \widetilde{B}) = \left\{ A, B : \ (A - \widetilde{A})^2 + (B - \widetilde{B})^2 \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{25}$$

The requirement is that the satisfaction be at least $S_{\rm c}$. Derive the robustness of decision q.

10. **Hybrid uncertainty: uncertain probabilistic task time.** (p.17) The probability that the project will be completed within the critical duration $t_{\rm c}$ is:

$$P = \frac{t_{\rm c}}{t_{\rm c} + uq} \tag{26}$$

where q is the time required to set up the project before actual work begins, and controlled by the manager, and u is uncertain:

$$\mathcal{U}(\alpha, \widetilde{u}) = \{ u : \ |u - \widetilde{u}| \le \alpha \widetilde{u} \}, \quad \alpha \ge 0 \tag{27}$$

The customer demands that the task complete within duration $t_{\rm c}$ with probability no less than $P_{\rm c}$. Derive an explicit algebraic expression for the robustness of q.

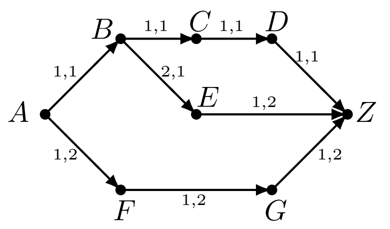

Figure 1: Transportation network for problem 11.

11. **Transportation network.** (p.17) A transportation network is specified by the directed graph in fig. 1. You must travel from node A to node Z along one of the three paths:

Path 1:	$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow Z$
Path 2:	$A \longrightarrow B \longrightarrow E \longrightarrow Z$
Path 3:	$A \longrightarrow F \longrightarrow G \longrightarrow Z$

The estimated transit time between nodes i and j is $\widetilde{t}_{ij}$, while the unknown true transit time is t_{ij}. The info-gap model for transit-time uncertainty is:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \ |t_{ij} - \widetilde{t}_{ij}| \le \alpha w_{ij} \widetilde{t}_{ij}, \ \text{for all} \ \ i, j \right\}, \quad \alpha \ge 0 \tag{28}$$

where w_{ij} is a non-negative known uncertainty weight. The values of $\widetilde{t}_{ij}$, w_{ij} appear along the edges of the graph in fig. 1. For instance, $\widetilde{t}_{EZ}, w_{EZ} = 1, 2$.

(a) It is required to get from A to Z in a duration no longer than $t_{\rm c}$. Derive expressions for the robustness to transit-duration uncertainty for each of the three paths.

(b) For what values of $t_{\rm c}$ will you prefer each of the three paths, based on the criterion of maximizing the robustness and satisficing the total transit time?

Budget item	In-House			Out-Source		
	$\widetilde{c}_i$	w_i	d_i	$\widetilde{c}_i$	w_i	d_i
1	1.5	1.0	1.0	1.3	1.3	1.0
2	2.7	0.7	0.7	2.2	1.0	0.9
3	1.4	1.2	0.8	1.5	1.3	0.8

Table 1: Data for problem 12.

12. **In-house or out-source?** (p.18) Our firm is planning a project which can either be implemented in-house, or purchased by out-sourcing. The project involves three main budget items, whose expenses are incurred at different stages of the project, and thus entail different financing costs. The total expense of the project is:

$$E(c) = \sum_{i=1}^{3} d_i c_i \tag{29}$$

where c_i is the cost of stage i and d_i is the discount factor for that stage. We require that the expense not exceed the budget, B.

The estimated costs $\widetilde{c}_i$, cost-uncertainties w_i, and discount factors d_i for the three budget items, for in-house and out-source options, are listed in table 1. Use a fractional-error info-gap model:

$$\mathcal{U}(\alpha, \widetilde{c}) = \left\{ c: \ |c_i - \widetilde{c}_i| \le w_i \widetilde{c}_i \alpha, \ i = 1, 2, 3 \right\}, \quad \alpha \ge 0 \tag{30}$$

Study the choice between in-house and out-source options, as a function of the total budget B. Which is preferable, as a function of the total budget constraint?

13. **Failure probability and financial loss.** We are designing a production system which is subject to failures (cracks, leaks, down times, etc.). The severity of failure is the random variable x, where large x means large failure. We must choose between two alternative systems. The less expensive system, $i = 1$, is more prone to failure, but can be implemented with greater redundancy so the financial loss due to failure is lower. The more expensive system, $i = 2$, is less failure-prone but failures are more disruptive and hence more expensive.

The best estimate of the probability density function for failure with system i is exponential:

$$\widetilde{p}_i(x) = \lambda_i \mathrm{e}^{-\lambda_i x}, \quad x \ge 0 \tag{31}$$

$0 < \lambda_1 < \lambda_2$, expressing the fact that system 1 is more prone to failure. The estimated pdf, $\widetilde{p}_i(x)$, is highly uncertain, especially far out on the tail (large failure), and the true pdf, $p_i(x)$, is unknown. An info-gap model for uncertainty in the pdf is:

$$\mathcal{U}(\alpha, \widetilde{p}_i) = \left\{ p(x): \ p(x) \ge 0, \ \int_0^\infty p(x)\,\mathrm{d}x = 1, \ |p(x) - \widetilde{p}_i(x)| \le \alpha \widetilde{p}_i(x) \right\}, \quad \alpha \ge 0 \tag{32}$$

The financial loss resulting from failure of severity x, with option i, is:

$$L_i(x) = c_i x^2 \tag{33}$$

$0 < c_1 < c_2$ expresses the fact that failures in system 1 are less costly than in system 2.

We require that the probability of loss exceeding L_c be less than P_c. It would be wonderful if the probability of loss exceeding L_w (which is less than L_c) were less than P_w (which is less than P_c). Derive robustness and opportuneness functions for the two systems, and discuss the implications for choosing between the systems.

Simplification: assume that $\sqrt{L_\mathrm{c}/c_i}$ is greater than the median of $\widetilde{p}_i(x)$.

14. **Value of managerial attention.** (p.20) The value of managerial attention usually increases in time as the project approaches completion at time T. However, the value function is highly uncertain. The estimated value of attention at time t is:

$$\widetilde{v}(t) = v_1 t, \quad 0 \le t \le T \tag{34}$$

where v_1 is known and positive. However, the uncertainty in the true value, which may in fact be negative and need not always be positive or monotonic, is represented by the following info-gap model:

$$\mathcal{U}(\alpha) = \left\{ v(t): \ \left| \frac{v(t) - \widetilde{v}(t)}{v_1 T} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{35}$$

The manager will plan his degree of attention to the project in question, as specified by a "focus function" $f(t)$ which expresses a degree of managerial attention devoted to the project. The utility of focus $f(t)$ at time t depends on the value of attention at that time. The overall value of the manager's focus function is:

$$R(f, v) = \int_0^T f(t) v(t) \, \mathrm{d}t \tag{36}$$

We will consider a simple focus function of the form:

$$f(t) = f_0 t \tag{37}$$

where the manager must choose the value of f_0.

Evaluate the robustness of focus f_0 given critical value R_c.

15. **Product development effort.** A firm is developing a new product, whose anticipated reward is $\widetilde{r}_1$ per unit of effort, E, which is invested. Thus the anticipated reward from developing product 1 is:

$$g(E) = \widetilde{r}_1 E \tag{38}$$

This same effort could be invested in an alternative product whose anticipated reward is $\widetilde{r}_2$ per unit of effort, E, which is invested, where $\widetilde{r}_2 > \widetilde{r}_1$. However, a sunk cost of c would be lost in abandoning the first product and moving to the second product. The sunk cost is small, so $\widetilde{r}_2 - c > \widetilde{r}_1$. The anticipated reward from developing product 2 is:

$$g(E) = \widetilde{r}_2 E - c \tag{39}$$

The fractional errors of the estimates, $\widetilde{r}_1$ and $\widetilde{r}_2$, are unknown. Also, the project fails if the reward is less g_c.

(a) Derive the robustness function for choosing to develop each product with effort E.

(b) The firm will go out of business if it earns less than g_c. What is the greatest value of g_c at which the firm should prefer staying with product 1?

(c) Now consider product 1 again, but with a different info-gap model for uncertainty in the reward:

$$g(E) = \widetilde{r}_1 E + \sum_{m=2}^{N} r_m E^m \tag{40}$$

$\widetilde{r}_1$ is known but the coefficients r_m are uncertain. Let $\rho = (r_2, \ldots, r_N)^T$ and $\epsilon = (E^2, \ldots, E^N)^T$. Use an ellipsoidal info-gap model:

$$\mathcal{U}(\alpha) = \left\{ \rho : \ \rho^T W \rho \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{41}$$

where W is a known, real, symmetric, positive definite matrix. Derive the robustness function.

16. **A new product.** The quantity demanded, price per unit and total production cost, as a function of time, are $q(t)$, $p(t)$ and $c(t)$. The time horizon is $t \in [0, 1]$. The net income is:

$$R = \int_0^1 [q(t)p(t) - c(t)]\, \mathrm{d}t \tag{42}$$

We have estimates of these functions, which are constant in time: $\widetilde{q}$, $\widetilde{p}$ and $\widetilde{c}$. The actual time-varying values are uncertain, but we have the following understanding:

- Quantity demanded is substantially higher in mid-season, maybe by about 50%, but this is a new product so we don't know very well.
- Sale price is proportional to quantity demanded, $p(t) = \kappa q(t)$. The value of κ is around 2.5 plus or minus about 0.5, but it can fluctuate greatly due to fads and fashions or competition.
- Production costs are fairly stable compared to the other factors.

(a) Construct an info-gap model for uncertainty in the functions of this model.

(b) Use this info-gap model to construct the robustness function, given the requirement that the income exceed R_{c}.

(c) Use this info-gap model to construct the opportuneness function, given the aspiration that the income exceed R_{w}.

17. **Budgeting uncertain costs** (p.22). Two budget items have estimated costs $\widetilde{c}_1 = 10 \pm 4$ and $\widetilde{c}_2 = 6 \pm 2$. The errors are estimates based on judgment and past experience. In addition, experience has shown that when expenditure on one of these items increases by \$1, the expenditure on the other tends to decrease by about \$0.2. Thus the covariance is roughly estimated as $-\$2$.

(a) How would you quantify the uncertainty in the expenses?

(b) What budget do you need?

18. **Managerial attention** (p.22). A manager must allocate time between two tasks. The time allocated to the ith task is t_i, and the total time available is T. The reward from task i resulting from this allocation is:

$$r_i(t_i) = \frac{1}{\lambda_i t_i} \tag{43}$$

The total reward, $r(t)$, is the sum of the two single-task rewards. We require that the total reward exceed r_{c}.

(a) The coefficients λ_i are uncertain, as expressed by the following info-gap model:

$$\mathcal{U}(\alpha) = \left\{ \lambda_i :\ \lambda_i > 0,\ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{\widetilde{\lambda}_i} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{44}$$

Derive an expression for the robustness.

(b) Continuing part (a), find the time-allocation which maximizes the robustness, given the constraint on the total time available and assuming that each task must receive at least a duration $0 < \tau \ll T/2$.

(c) Now change the info-gap model of eq.(44) to include information which distinguishes between the fractional errors of $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$:

$$\mathcal{U}(\alpha) = \left\{ \lambda_i :\ \lambda_i > 0,\ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{s_i} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{45}$$

Derive an expression for the inverse of the robustness function and compare it with the inverse of the robustness function from part (a).

(d) Now change the info-gap model to the following envelope-bound model:

$$\mathcal{U}(\alpha) = \left\{ r_i(t_i): \ \left| \frac{r_i(t_i) - \widetilde{r}_i(t_i)}{\widetilde{r}_i(t_i)} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{46}$$

where $\widetilde{r}_i(t_i)$ is defined by eq.(43) with $\widetilde{\lambda}_i$. Derive the robustness function.

(e) Now change the info-gap model of eq.(46) to the following envelope-bound model:

$$\mathcal{U}(\alpha) = \left\{ r_i(t_i): \ \left| \frac{r_i(t_i) - \widetilde{r}_i(t_i)}{s_i(t_i)} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{47}$$

where $\widetilde{r}_i(t_i)$ is defined by eq.(43) with $\widetilde{\lambda}_i$. We allow negative reward (penalty). Derive the robustness function.

(f) Now consider a probabilistic version of this problem. Let $t_{\max}$ denote the larger of the two time allocations:

$$t_{\max} = \max[t_1, t_2] \tag{48}$$

Let us suppose that the total reward depends only on $t_{\max}$. Furthermore, our estimate of the pdf of the total reward is the following gamma distribution:

$$\widetilde{p}(r) = t_{\max}^2 r \mathrm{e}^{-r t_{\max}}, \quad r \ge 0 \tag{49}$$

Let $\widetilde{P}(r)$ denote the estimated cumulative distribution function for total reward.

The actual distribution of reward is uncertain, with pdf and cdf $p(r)$ and $P(r)$, respectively. In particular, we suspect that there is non-zero probability of negative reward (penalty) though we have no idea what the pdf for negative r is, though we believe that the shape of the gamma distribution describes the distribution of positive rewards. Thus we use the following info-gap model:

$$\mathcal{U}(\alpha) = \left\{ p(r): \ P(r < 0) = \alpha, \ p(r) = (1-\alpha)\widetilde{p}(r) \text{ for } r \ge 0 \right\}, \quad \alpha \ge 0 \tag{50}$$

We require that the probability of reward at least as large as r_{c} must be no less than P_{c}:

$$P(r \ge r_{\mathrm{c}}) \ge P_{\mathrm{c}} \tag{51}$$

Derive the robustness function for $r_{\mathrm{c}} > 0$.

(g) Let $t = (t_1, t_2)^T$ denote the column vector of task times and suppose that the total reward is:

$$r(t) = \rho^T t \tag{52}$$

where ρ is uncertain with info-gap model:

$$\mathcal{U}(\alpha) = \left\{ \rho: \ (\rho - \widetilde{\rho})^T W (\rho - \widetilde{\rho}) \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{53}$$

We require total reward no less than r_{c}. Derive the robustness function.