A Tutorial Introduction to Quantitative Finance

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And Now for Something Completely Different!

Outline

- Preliminaries: Introduction of terminology, problems studied, etc.
- Finite market models: Basic ideas in an elementary setting.
- Black-Scholes theory: What is it and why is it so (in-)famous?
- Some simple modifications of Black-Scholes theory
- What went wrong?
- Some open problems that would interest control theorists

Preliminaries

Some Terminology

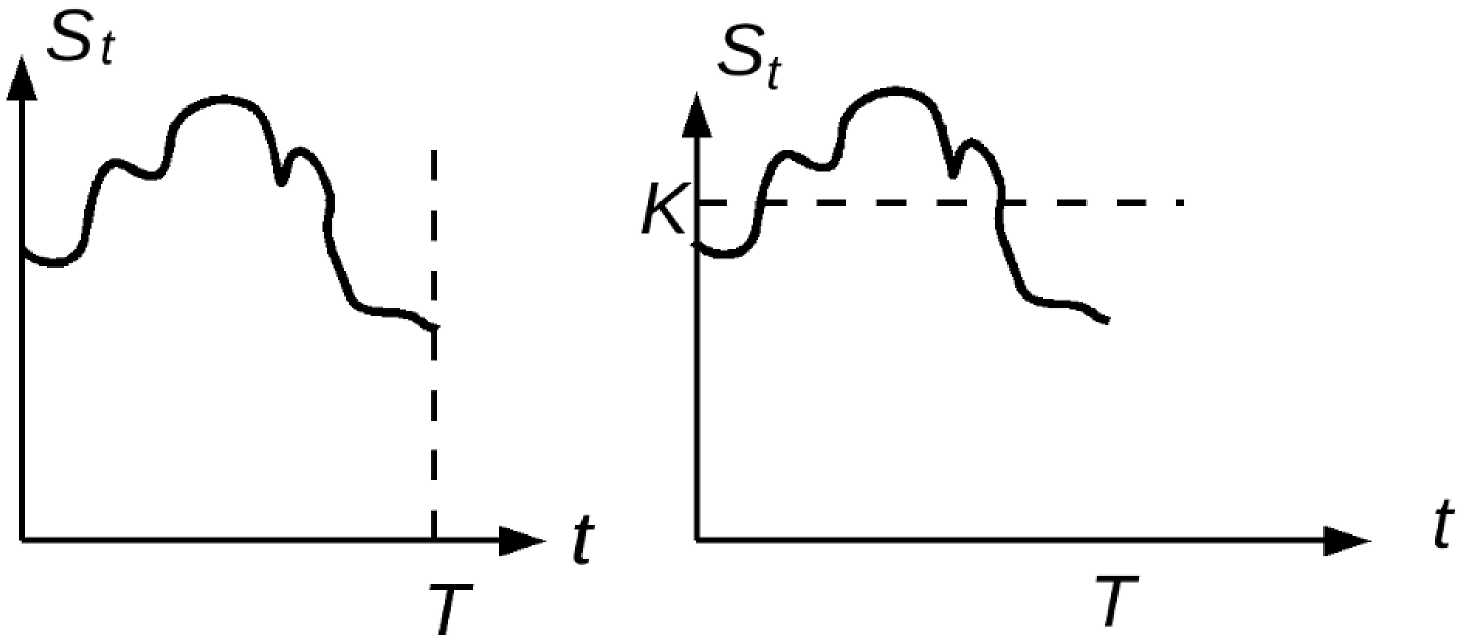
A market consists of a ‘safe’ asset usually referred to as a ‘bond’, together with one or more ‘uncertain’ assets usually referred to as ‘stocks’.

A ‘portfolio’ is a set of holdings, consisting of the bond and the stocks. We can think of it as a vector in $\mathbb{R}^{n+1}$ where n is the number of stocks. Negative ‘holdings’ correspond to borrowing money or ‘shorting’ stocks.

An ‘option’ is an instrument that gives the buyer the right, but not the obligation, to buy a stock a prespecified price called the ‘strike price’ K.

A ‘European’ option can be exercised only *at* a specified time T. An ‘American’ option can be exercised *at any time prior to* a specified time T.

European vs. American Options



The value of the European option is $\{S_T - K\}_+$. In this case it is worthless even though $S_t > K$ for some intermediate times. The American option has positive value at intermediate times but is worthless at time $t = T$.

The Questions Studied Here

- What is the minimum price that the seller of an option should be willing to accept?
- What is the maximum price that the buyer of an option should be willing to pay?
- How can the seller (or buyer) of an option ‘hedge’ (minimimze or even eliminate) his risk after having sold (or bought) the option?

Finite Market Models

One-Period Model

Many key ideas can be illustrated via ‘one-period’ model.

We have a choice of investing in a ‘safe’ bond or an ‘uncertain’ stock.

$B(0)$ = Price of the bond at time $T = 0$. It increases to $B(1) = (1+r)B(0)$ at time $T = 1$.

$S(0)$ = Price of the stock at time $T = 0$.

$$S(1) = \begin{cases} S(0)u & \text{with probability } p, \\ S(0)d & \text{with probability } 1-p. \end{cases}$$

Assumption: $d < 1 + r < u$; otherwise problem is meaningless!

Rewrite as $d' < 1 < u'$, where $d' = d/(1+r), u' = u/(1+r)$.

Options and Contingent Claims

An ‘option’ gives the buyer the right, but not the obligation, to buy the stock at time $T = 1$ at a predetermined strike price K. Again, assume $S(0)d < K < S(0)u$.

More generally, a ‘contingent claim’ is a random variable X such that

$$X = \begin{cases} X_u & \text{if } S(1) = S(0)u, \\ X_d & \text{if } S(1) = S(0)d. \end{cases}$$

To get an option, set $X = \{S(1) - K\}_+$. Such instruments are called ‘derivatives’ because their value is ‘derived’ from that of an ‘underlying’ asset (in this case a stock).

Question: How much should the seller of such a claim charge for the claim at time $T = 0$?

An Incorrect Intuition

View the value of the claim as a random variable.

$$X = \begin{cases} X_u & \text{with probability } p_u = p, \\ X_d & \text{with probability } p_d = 1 - p. \end{cases}$$

So

$$E[X, \mathbf{p}] = (1 + r)^{-1}[pX_u + (1 - p)X_d].$$

Is this the ‘right’ price for the contingent claim?

NO! The seller of the claim can ‘hedge’ against future fluctuations of stock price by using a part of the proceeds to buy the stock himself.

The Replicating Portfolio

Build a portfolio at time $T = 0$ such that its value *exactly matches* that of the claim at time $T = 1$ *irrespective* of stock price movement.

Choose real numbers a and b (investment in stocks and bonds respectively) such that

$$aS(0)u + bB(0)(1+r) = X_u, aS(0)d + bB(0)(1+r) = X_d,$$

or in vector-matrix notation

$$[a \;\; b] \begin{bmatrix} S(0)u & S(0)d \\ B(0)(1+r) & B(0)(1+r) \end{bmatrix} = [X_u \;\; X_d].$$

This is called a ‘replicating portfolio’. The unique solution is

$$[a \;\; b] = [X_u \;\; X_d] \begin{bmatrix} S(0)u & S(0)d \\ B(0)(1+r) & B(0)(1+r) \end{bmatrix}^{-1}.$$

Cost of the Replicating Portfolio

So how much money is needed at time $T = 0$ to implement this replicating strategy? The answer is

$$\begin{aligned} c &= [a \;\; b] \begin{bmatrix} S(0) \\ B(0) \end{bmatrix} \\ &= [X_u \;\; X_d] \begin{bmatrix} S(0)u & S(0)d \\ B(0)(1+r) & B(0)(1+r) \end{bmatrix}^{-1} \begin{bmatrix} S(0) \\ B(0) \end{bmatrix} \\ &= (1+r)^{-1} [X_u \;\; X_d] \begin{bmatrix} q_u \\ q_d \end{bmatrix}, \end{aligned}$$

where with $u' = u/(1+r), d' = d/(1+r)$, we have

$$\mathbf{q} := \begin{bmatrix} q_u \\ q_d \end{bmatrix} = \begin{bmatrix} u' & d' \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-d'}{u'-d'} \\ \frac{u'-1}{u'-d'} \end{bmatrix}$$

Note that $q_u, q_d > 0$ and $q_u + q_d = 1$. So $\mathbf{q} := (q_u, q_d)$ is a probability distribution on $S(1)$.

Martingale Measure: First Glimpse

Important point: $\mathbf{q}$ depends only on the returns u, d, and not on the *associated probabilities* $p, 1-p$.

Moreover,

$$E[(1+r)^{-1}S(1), \mathbf{q}] = S(0)u'\frac{1-d'}{u'-d'} + S(0)d'\frac{u'-1}{u'-d'} = S(0).$$

Under this synthetic distribution, the discounted expected value of the stock price at time $T = 1$ equals $S(0)$. Hence $\{S(0), (1+r)^{-1}S(1)\}$ is a ‘martingale’ under the synthetic probability distribution $\mathbf{q}$.

Thus

$$c = (1+r)^{-1}[X_u \ \ X_d]\begin{bmatrix} q_u \\ q_d \end{bmatrix}$$

is the *discounted expected value* of the contingent claim X under the martingale measure $\mathbf{q}$.

Arbitrage-Free Price of a Claim

Theorem: The quantity

$$c = (1+r)^{-1}[X_u \;\; X_d]\begin{bmatrix} q_u \\ q_d \end{bmatrix}$$

is the *unique arbitrage-free price* for the contingent claim.

Suppose someone is ready to pay $c' > c$ for the claim. Then the seller collects c', invests $c' - c$ in a risk-free bond, uses c to implement replicating strategy and settle claim at time $T = 1$, and pockets a risk-free profit of $(1+r)(c' - c)$. This is called an 'arbitrage opportunity'.

Suppose someone is ready to sell the claim for $c' < c$. Then the *buyer* can make a risk-free profit.

Multiple Periods: Binomial Model

The same strategy works for multiple periods; this is called the **binomial model**.

Bond price is deterministic:

$$B_{n+1} = (1 + r_n)B_n, n = 0, \ldots, N - 1.$$

Stock price can go up or down: $S_{n+1} = S_n u_n$ or $S_n d_n$.

There are 2^N possible sample paths for the stock, corresponding to each $\mathbf{h} \in \{u, d\}^N$. For each sample path $\mathbf{h}$, at time N there is a payout $X_{\mathbf{h}}$.

Claim becomes due only *at the end* of the time period (European contingent claim). In an 'American' option, the buyer chooses the time of exercising the option.

Define $d'_n = d_n/(1+r_n), u'_n = u_n/(1+r_n)$,

$$q_{u,n} = \frac{1-d'_n}{u'_n - d'_n}, q_{d,n} = \frac{u'_n - 1}{u'_n - d'_n;}.$$

Introduce a modified stochastic process

$$\mathsf{S}_{n+1} = \begin{cases} \mathsf{S}_n u'_n & \text{with probability } q_{u,n}, \\ \mathsf{S}_n d'_n & \text{with probability } q_{d,n}. \end{cases}$$

Then

$$E\{(1+r_n)^{-1}\mathsf{S}_{n+1} | \mathsf{S}_n, \mathsf{S}_{n-1}, \ldots, \mathsf{S}_0\} = \mathsf{S}_n, \text{ for } n = 0, \ldots, N-1.$$

Thus the discounted process

$$\left\{ \prod_{i=0}^{n-1} (1+r_i)^{-1} \mathsf{S}_n \right\}$$

where the empty product is taken as one, is a martingale.

Replicating Strategy for N Periods

We already know to replicate over one period. Extend argument to N periods.

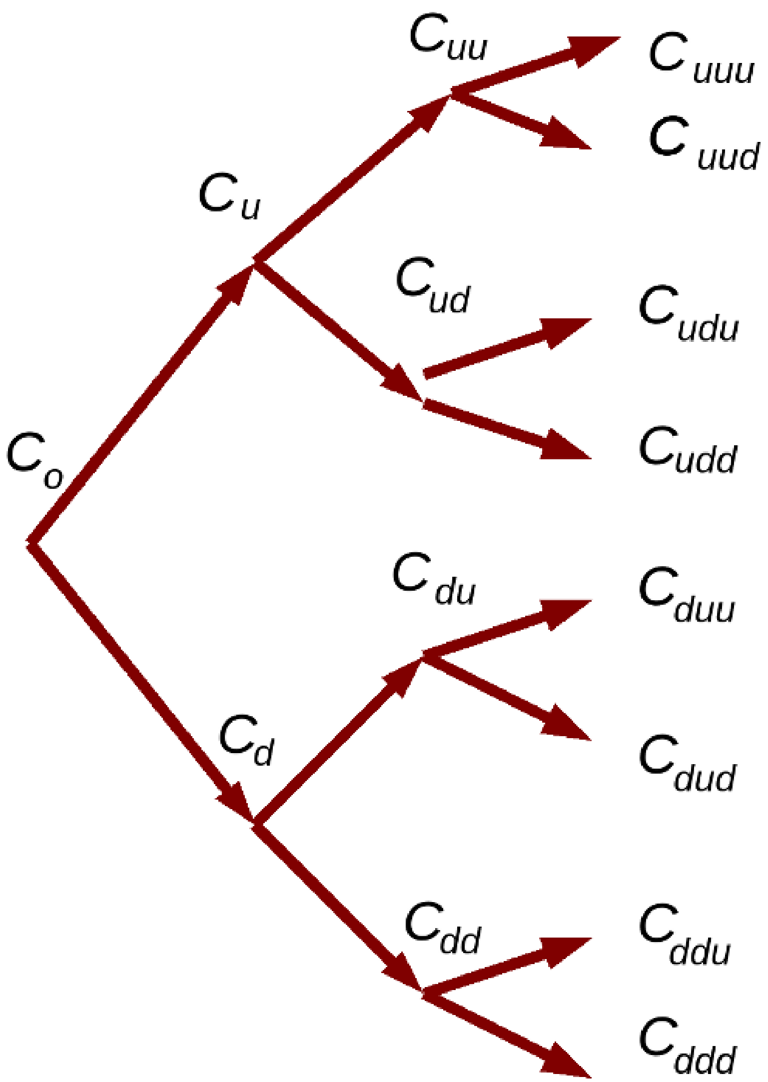
Suppose $\mathbf{j} \in \{u, d\}^{N-1}$ is the set of stock price transitions up to time $N-1$. So now there are only two possibilities for the final sample path: $\mathbf{j}u$ and $\mathbf{j}d$, and only two possible payouts: $X_{\mathbf{j}u}$ and $X_{\mathbf{j}d}$. Denote these by $c_{\mathbf{j}u}$ and $c_{\mathbf{j}d}$ respectively. We already know how to compute a cost $c_{\mathbf{j}}$ and a replicating portfolio $[a_{\mathbf{j}} \;\; b_{\mathbf{j}}]$ to replicate this claim, namely:

$$c_{\mathbf{j}} = (1 + r_{n-1})^{-1} \left(c_{\mathbf{j}u} q_{u,n} + c_{\mathbf{j}d} q_{d,n} \right).$$

$$[a_{\mathbf{j}} \;\; b_{\mathbf{j}}] = [c_{\mathbf{j}u} \;\; c_{\mathbf{j}d}] \left[\begin{array}{ll} S_{\mathbf{j}} u_{\mathbf{j}} & S_{\mathbf{j}} d_{\mathbf{j}} \\ B_{\mathbf{j}}(1 + r_{N-1}) & B_{\mathbf{j}}(1 + r_{N-1}) \end{array} \right]^{-1}.$$

Do this for *each* $\mathbf{j} \in \{u, d\}^{N-1}$. So if we are able to replicate each of the 2^{N-1} payouts $c_{\mathbf{j}}$ at time $T = N - 1$, then we know how to replicate each of the 2^N payouts at time $T = N$.

Repeat backwards until we reach time $T = 0$. Number of payouts decreases by a factor of two at each time step.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Arbitrage-Free Price for Multiple Periods

Recall earlier definitions:

$$d'_n = \frac{d_n}{1+r_n}, u'_n = \frac{u_n}{1+r_n}, q_{u,n} = \frac{1-d'_n}{u'_n - d'_n}, q_{d,n} = \frac{u'_n - 1}{u'_n - d'_n;}.$$

Now define

$$q_{\mathbf{h}} = \prod_{n=0}^{N-1} q_{h,n}, \ \forall \mathbf{h} \in \{u, d\}^N,$$

$$c_0 := \left[\prod_{n=0}^{N-1} (1 + r_n)\right]^{-1} \sum_{\mathbf{h} \in \{u,d\}^N} X_{\mathbf{h}} q_{\mathbf{h}}.$$

c_0 is the expected value of the claim $X_{\mathbf{h}}$ under the synthetic distribution $\{q_{\mathbf{h}}\}$ that makes the discounted stock price a martingale. Moreover, c_0 is the unique arbitrage-free price for the claim.

Replicating Strategy in Multiple Periods

Seller of claim receives an amount c_0 at time $T = 0$ and invests a_0 in stocks and b_0 in bonds, where

$$[a_0 \;\; b_0] = [c_u \;\; c_d] \begin{bmatrix} S_0 u_0 & S_0 d_0 \\ B_0(1 + r_0) & B_0(1 + r_0) \end{bmatrix}^{-1}.$$

Due to replication, at time $T = 1$, the portfolio is worth c_u if the stock goes up, and is worth c_d if the stock goes down. At time $T = 1$, adjust the portfolio according to

$$[a_1 \;\; b_1] = [c_{i_0 u} \;\; c_{i_0 d}] \begin{bmatrix} S_1 u_1 & S_1 d_1 \\ B_1(1 + r_1) & B_1(1 + r_1) \end{bmatrix}^{-1},$$

where $i_0 = u$ or d as the case may be. Then repeat.

Important note: This strategy is **self-financing:**

$$a_0 S_1 + b_0 B_1 = a_1 S_1 + b_1 B_1$$

whether $S_1 = S_0 u_0$ or $S_1 = S_0 d_0$ (i.e. whether the stock goes up or down at time $T = 1$). This property has no analog in the one-period case.

It is also *replicating* from that time onwards.

Observe: Implementation of replicating strategy requires reallocation of resources N times, once at each time instant.

Black-Scholes Theory

Continuous-Time Processes: Black-Scholes Formula

Take 'limit' at time interval goes to zero and $N \to \infty$; binomial asset price movement becomes geometric Brownian motion:

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right], t \in [0, T],$$

where W_t is a standard Brownian motion process. μ is the 'drift' of the Brownian motion and σ is the volatility.

Bond price is deterministic: $B_t = B_0 e^{rt}$. Claim is European and a simple option: $X_T = \{S_T - K\}_+$.

What we *can* do: Make μ, σ, r functions of t and not constants.

What we *cannot* do: Make σ, r *stochastic!* (Stochastic μ is OK.)

Theorem (Black-Scholes 1973): The unique arbitrage-free option price is

$$C_0 = S_0 \Phi\left(\frac{\log(S_0/K^*)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - K^*\Phi\left(\frac{\log(S_0/K^*)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right),$$

where

$$\Phi(c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-u^2/2}du$$

is the Gaussian distribution function, and $K^* = e^{-rT}K$ is the discounted strike price.

Black-Scholes PDE

Consider a general payout function $e^{rT}\psi(e^{-rT}x)$ to the buyer if $S_T = x$ (various exponentials discount future payouts to $T = 0$). Then the unique arbitrage-free price is given by

$$C_0 = f(0, S_0),$$

where f is the solution of the PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0,\ \forall (t, x) \in (0, T) \times (0, \infty),$$

with the boundary condition

$$f(T, x) = \psi(x).$$

No closed-form solution in general (but available if $\psi(x) = (x - K^*)_+$).

Replicating Strategy in Continuous-Time

Define

$$C_t^* = C_0 + \int_0^t f_x(s, S_s^*) dS_s^*, t \in (0, T),$$

where the integral is a stochastic integral, and define

$$\alpha_t = f_x(t, S_t^*), \beta_t = C_t^* - \alpha_t S_t^*.$$

Then hold α_t of the stock and β_t of the bond at time t.

Observe: Implementation of self-financing fully replicating strategy requires *continuous trading*.

Some Simple Modifications of Black-Scholes Theory

Extensions to Multiple Assets

Binomial model extends readily to multiple assets.

Black-Scholes theory extends to the case of multiple assets of the form

$$S_t^{(i)} = S_0^{(i)} \exp\left[\left(\mu^{(i)} - \frac{1}{2}[\sigma^{(i)}]^2\right)t + \sigma W_t^{(i)}\right], t \in [0, T],$$

where $W_t^{(i)}, i = 1, \ldots, d$ are (possibly correlated) Brownian motions.

Analog of Black-Scholes PDE: $C_0 = f(0, S_0^{(1)}, \ldots, S_0^{(d)})$ where f satisfies a PDE. But no closed-form solution for f in general.

American Options

An ‘American’ option can be exercised at any time *up to and including* time T.

So we need a ‘super-replicating’ strategy: The value of our portfolio must *equal or exceed* the value of the claim at all times.

If $X_t = \{S_t - K\}_+$, then both price and hedging strategy are same as for European claims.

Very little known about pricing American options in general. Theory of ‘optimal time to exercise option’ is very deep and difficult.

Sensitivities and the 'Greeks'

Recall $C_0 = f(0, S_0)$ is the correct price for the option under Black-Scholes theory. (We *need not* assume the claim to be a simple option!)

$$\Delta = \frac{\partial C_0}{\partial S_0}, \Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{\partial^2 C_0}{\partial S_0^2},$$

$$\text{Vega} = \nu = \frac{\partial C_0}{\partial \sigma}, \theta = -\frac{\partial f(t, X_t)}{\partial t}, \rho = \frac{\partial f(t, X_t)}{\partial r}.$$

'Delta-hedging': A strategy such that $\Delta = 0$ – return is insensitive to initial stock price (to first-order approximation).

'Delta-gamma-hedging': A strategy such that $\Delta = 0, \Gamma = 0$ – return is insensitive to initial stock price (to second-order approximation).

What Went Wrong?

What Went Wrong?

My view: The current financial debacle owes *very little* to poor modeling of risks. Most significant factors were:

- Complete abdication of oversight responsibility by US government – led to over-leveraging and massive conflicts of interests
- OTC trading of complex instruments – made 'price discovery' impossible
    - ⇒ Net outstanding derivative positions: *$ 760 trillion!*
    - ⇒ U.S. Annual GDP: $ 13 trillion, World's: $ 60 trillion
    - ⇒ 90% of derivatives traded OTC – No regulation whatsoever!
- 'One-way' rewards for traders: Heads the traders win – tails the depositors and shareholders lose
- Real bonuses paid on virtual profits, and so on

Please read full paper for elaboration.

Some Open Problems That Would Interest Control Theorists

Some Open Problems

- Incomplete markets: Replication is impossible
- Multiple martingale measures: Which one to choose?
- Computing the ‘greeks’ when there are no closed-form formulas
- Alternate models for asset price movement: Why geometric Brownian motion?

Incomplete Markets

Consider the one-period model where the asset takes *three*, not two, possible values.

$$S(1) = \begin{cases} S(0)u & \text{with probability } p_u, \\ S(0)m & \text{with probability } p_m, \\ S(0)d & \text{with probability } p_d, \end{cases}$$

To construct a replicating strategy, we need to choose a, b such that

$$[a \;\; b] \begin{bmatrix} S(0)u & S(0)m & S(0)d \\ (1+r)B(0) & (1+r)B(0) & (1+r)B(0) \end{bmatrix} = [X_u \;\; X_m \;\; X_d].$$

No solution in general – replication is impossible!

Martingale measures

Let us try to find a synthetic measure $\mathbf{q} = (q_u, q_m, q_d)$ so that

$$E[(1+r)^{-1}S(1), \mathbf{q}] = S(0).$$

Need to choose q_u, q_m, q_d such that

$$q_u + q_m + q_d = 1, q_u u + q_m m + q_d d = 1 + r.$$

Infinitely many martingale measures!

Is there a relationship?

Finite Market Case: DMW Theorem

Consider d assets over N time instants, each assuming values in $\mathbb{R}$. We allow infinitely many values for each asset, correlations, etc. Let $\tilde{P}$ denote the law of these assets (law over $\mathbb{R}^{dN}$).

Theorem (Dalang-Morton-Willinger): The following statements are equivalent:

- There exists a probability measure $\tilde{Q}$ on $\mathbb{R}^{dN}$ that is equivalent to $\tilde{P}$ such that, under $\tilde{Q}$, the process $\{S_n\}_{n=0}^{N}$ is a martingale.
- There does not exist an arbitrage opportunity.

Moreover, if either of these equivalent conditions holds, then it is possible to choose the measure $\tilde{Q}$ in such a way that the Radon-Nikodym derivative $(d\tilde{Q}/d\tilde{P})$ is bounded.

Existence of a Replicating Strategy

Define

$$V_-(X) := \min_{\tilde{Q}\in\mathcal{M}} E[X,\tilde{Q}], V_+(X) := \max_{\tilde{Q}\in\mathcal{M}} E[X,\tilde{Q}],$$

where $\mathcal{M}$ is the set of martingale measures. Then *every price* in $[V_-(X), V_+(X)]$ is arbitrage-free.

Note: $V_-(X)$ is the maximum that the buyer of the claim would (normally) be willing to pay. Similarly, $V_+(X)$ is the minimum that the seller of the claim would (normally) be willing to accept.

Theorem: A European contingent claim with the payout function X is replicable if and only if $V_-(X) = V_+(X)$.

Modulo technical conditions, replicability ≈ unique martingale measure.

Minimum Relative Entropy Martingage Measures

In incomplete markets ($\mathcal{M}$ is infinite), choose $\tilde{Q}$ to minimize the *relative entropy*:

$$\tilde{Q}^* = \arg \min_{\tilde{Q} \in \mathcal{M}} H(\tilde{Q} \parallel \tilde{P}),$$

where

$$H(\tilde{Q} \parallel \tilde{P}) = \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i}.$$

Choose martingale measure that is ‘closest’ to the real-world law $\tilde{P}$.

$\tilde{Q}^*$ has a nice interpretation in terms of maximizing utility. Since $\mathcal{M}$ is a convex set and $H(\tilde{Q} \parallel \tilde{P})$ is convex in $\tilde{Q}$, this is a convex optimization problem.

Computing the Greeks

Even for simple geometric Brownian motion models, no closed form for fair price $C_0 = f(0, S_0)$ unless claim X is a simple option. So how to compute the 'greeks'?

One possibility: Evaluate fair price(s) using stochastic (Monte Carlo) simulation; then differentiate numerically. Absolutely fraught with numerical instabilities and sensitivities.

Answer: Malliavin calculus.

Originally developed for Brownian motion, now extented to arbitrary Lévy processes.

Alternate Models for Asset Prices

Geometric Lévy processes: Assume

$$S_t = S_0 \exp(L_t),$$

where $\{L_t\}$ is a Lévy process.

A Lévy process is the most general process with independent increments. For all practical purposes, a Lévy process consists of a Wiener process (Brownian motion) plus a countable number of ‘jumps’.

Sad fact: Replication is possible *only* for geometric Brownian motion asset prices!

Ergo: GBM (Geometric Brownian Motion) is the only asset price model for which there exists a unique martingale measure and the theory is very ‘pretty’.

Unfortunately, real world distributions are *quite far* from Gaussian, in the range $[\mu - 10\sigma, \mu + 10\sigma]$. Moreover, they are also ‘skewed’ in the positive direction.

A tractable theory of pricing claims with real world probabilities is still waiting to be discovered.

Conclusions

Quantitative finance has definitely made pricing and trading more ‘scientific’, but several poorly understood issues still remain:

- Irrational behavior
- Non-equilibrium economics (e.g. incomplete information)
- Correlated increments in asset price returns
- Effects of alternate asset price models on option pricing and/or trading strategies.

In short, what is unknown dwarfs what is known.

Thank You!