Quantum versus classical dynamics in a periodically driven anharmonic oscillator

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Numerical studies of the classical and quantum dynamics of a periodically driven anharmonic oscillator show that the *only* quasienergy states that are exponentially localized in the field-free energy space are those that are located in the regular region of classical phase space. All the quasienergy states located in the bounded chaotic region in phase space are extended states and do *not* show the *strong* quantum limitation of chaos due to the Anderson localization mechanism, which is characteristic of kicked systems.

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Quantum localization, suppression, and limitation of chaos [1] was obtained in many numerical studies of classical and quantum dynamics of periodically kicked systems such as a planar rotor [2–7], surface-state electrons [8], and hydrogen atoms in monochromatic fields [9,10]. In the kicked-rotor systems the strong quantum limitation of chaos is reflected in the exponential localization of the quasienergy states in the field-free energy space (or in angular momentum space), when the system was off resonance and the ratio between the rotor and the driving frequency is irrational. This quantum effect was found to be analogous to the Anderson model of localization in a one-dimensional lattice. As an experimental test for this mechanism the rotational excitation of PbTe or CsI molecules by periodic microwave pulses was proposed [11].

Our numerical studies of the dynamics of a driven anharmonic oscillator show a behavior that is markedly different from the kicked systems. Chaos is definitely *not* suppressed by the Anderson localization mechanism. The possible weak quantum localization of chaos in the driven anharmonic oscillator and its mechanism is currently under study [12]. The Hamiltonian of our model is a forced quartic oscillator

$$H(t)=\frac{p^2}{2m}+bx^4-f_0x\cos(\omega t)\ . \qquad (1)$$

Note that this is a special case of the classical Duffing equation, which was shown to be chaotic (see, e.g., Ref. [13]). It is also a special case of the linearly forced classical x^q oscillator studied in Ref. [14]. Not all of the four parameters appearing Eq. (1) are independent; in fact, three of them can be removed by simple rescaling. We present results for the parameter values $m=1$, $b=\frac{1}{4}$, $f_0=\frac{1}{2}$, and $\omega=1$. The classical phase space at $t=nT$ ($T=2\pi$ is the period of the driving force) represented in Fig. 1 shows a large *bounded* chaotic zone with a seemingly completely regular region embedded in it. As already discussed in Ref. [14], the transition between the inner chaotic and the outer regular regime is extremely sharp. This phenomenon is quite typical for all parameter values ($b>0$), whereas the details of the stability regions embedded in the chaotic sea vary with the parameters. Although the energy of the field-free anharmonic oscillator H_0 is not conserved in the presence of the external field, we see clearly from Fig. 1 that completely regular behavior is found when $H_0\geq 4$.

Numerical integration provides a phase space of $\oint p\,dx=2.25$ of the inner regular island region. The total phase-space area of the chaotic region including the embedded stability islands yields a value of $\oint p\,dx=10.1$.

The quantum Hamiltonian has the form of Eq. (1) with p and x replaced by operators $\hat{p}$ and $\hat{x}$. In the numerical computations we used a value of $\hbar=0.015$. The quasienergy states [15,16] are Floquet solutions of the time-dependent Schrödinger equation such that at times $t=nT$,

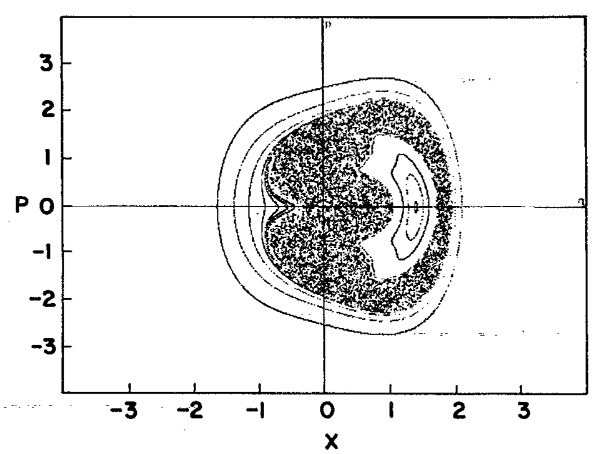

FIG. 1. Synoptic Poincaré section of classical phase space at $t=nT$ ($n=0,1,2,\ldots$). All points in the chaotic region result from a single classical trajectory.

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$$\psi_\alpha(x,nT)=e^{-in\theta_\alpha}\phi_\alpha(x,nT) , \tag{2}$$

$$|\psi_\alpha(x,nT)|=|\psi_\alpha(x,0)|, \quad n=0,1,2,\ldots \tag{3}$$

with $\theta_\alpha=(1/\hbar)\varepsilon_\alpha T$, where the ε_α are the quasienergies. The quasienergy states were obtained by diagonalization of the time evolution matrix for one optical cycle $T=2\pi$. The time evolution matrix U was obtained by solving the following time-dependent matrix problem

$$HU(0,t)=i\hbar\frac{dU(0,t)}{dt}, \quad t\in[0,T] \tag{4}$$

with initial condition $U(0,0)=1$ and

$$H_{ij}=\delta_{ij}E_i^{(0)}-f_0\langle\psi_i^{(0)}|x|\psi_j^{(0)}\rangle\cos(\omega t) , \tag{5}$$

where $E_i^{(0)}$ and $\{\psi^{(0)}\}$ are the 360 variational eigenvalues and eigenvectors of the field-free Hamiltonian H_0. The propagation of $U(0,t)$ was carried out by the Adams-Moulton predictor-corrector method [17]. The localized states were labeled by α in increasing order of the corresponding expectation value of H_0. As shown in Fig. 2, there are 113 states (with θ_α values covering the interval $[0,2\pi]$ almost uniformly) that provide almost degenerate states (i.e., $\langle\alpha|H_0|\alpha\rangle\approx 1$) as the field is suddenly turned off. The other states, $\alpha>113$, provide expectation values $\langle\alpha|H_0|\alpha\rangle$ that are very close to the highly excited energy spectrum of the free-field Hamiltonian H_0. These quasienergy states are associated with the outer regular zone shown in Fig. 1. The quasienergy states $\phi_\alpha(x,0)$ were variationally obtained in the basis of the field-free states $\phi_\alpha(x,0)=\sum_n C_{n\alpha}\psi_n^{(0)}(x,0)$. The variational linear parameters $\mathbf{C}_\alpha$ are eigenvectors of the time-evolution matrix

$$U(0,t=T)\mathbf{C}_\alpha=e^{-i\theta_\alpha}\mathbf{C}_\alpha . \tag{6}$$

The $\mathbf{C}_\alpha$ are the overlap integrals of the quasienergy states with the field-free states $\psi_n^{(0)}(x,0)$, and their behavior as function of N shows if they are localized or extended.

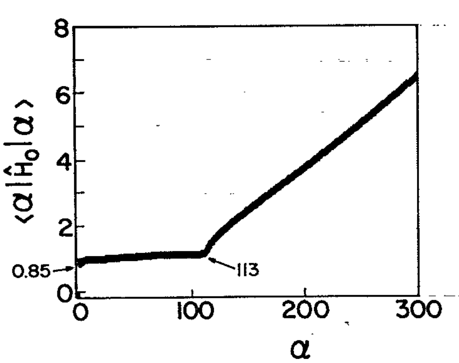

FIG. 2. Expectation values of the field-free Hamiltonian, $\langle\alpha|H_0|\alpha\rangle$, where $\{|\alpha\rangle\}$ are the variational quasienergy states at $t=nT$ $(n=0,1,2,\ldots)$.

The corresponding quantum analogue to the classical phase-space distribution can be obtained by calculating the Husimi distribution function

$$\Gamma_\alpha(x,p)=\int_{\text{all space}}\Phi_G^*(x')\gamma_\alpha(x',x'')\Phi_G(x'')dx'dx'' , \tag{7}$$

where Φ_G is a $\hbar$-width Gaussian centered at the point (x,p) in phase space. The density kernel $\gamma_\alpha(x',x')$ is in the one-dimensional case simply given by

$$\gamma_\alpha(x',x')=\sum_{n,m}C_{m\alpha}^*C_{n\alpha}\psi_n^{(0)}(x')\psi_m^{(0)*}(x'') . \tag{8}$$

$\Gamma_\alpha(x,p)$ provides the probability density of finding the quantum particle at phase space point (x,p).

In Fig. 3 we show some quasienergy states characterized by *exponential localization in the field-free energy space*. These states can be counted by the number of "nodes" in the $|C_{n\alpha}|$ distribution, which also appear in the corresponding plots of the Husimi phase-space density. The key point is that 24 exponentially localized states were found in the quantum calculations and *all* of them are located in the regular island region in the classical phase-space plot, Fig. 1. Note that a simple semiclassical quantization condition derived recently by Breuer and Holthaus [18] (compare also [19]),

$$\oint p\,dq=2\pi\hbar(n+\tfrac{1}{2}) \quad (n=0,1,2,\ldots) , \tag{9}$$

where the integral is taken over a closed invariant curve in the Poincaré section (see Fig. 1) yields a maximum number of 24 regular states within this regular island in agreement with the quantum results.

The nodal structure of the regular energy states, which is visible in the field-free energy space and in the Husimi distribution function, results from the fact that these states provide almost the same expectation value of H_0. The Husimi distribution of the $\psi_{73}^{(0)}$ eigenfunction of H_0 is predominantly concentrated in an annular phase-space region, which passes through the stable fixed point at the center of the classical regular island. The nodeless regular quasienergy state is therefore expected to be $\phi_{\alpha=7}\approx\psi_{73}^{(0)}$. However, in order to locate $\phi_{\alpha=7}$ only at $x>0$ (see the regular island in Fig. 1), we should break the x symmetry of the field-free eigenfunctions due to a quantum interference of about three to five states. That is,

$$\phi_{\alpha=7}\approx\sum_{k=-2}^{2}D_{k\alpha}\psi_{73+k}^{(0)}\equiv\tilde{\psi}_{73}^{(0)} , \tag{10}$$

where the $D_{k\alpha}$ are phase factors variationally obtained. This explains the noncusp behavior of $\ln|C_{n\alpha}|$ vs $E_n^{(0)}$ in Fig. 3. Since the regular quasienergy solutions provide about the same value of $\langle H_0\rangle$,

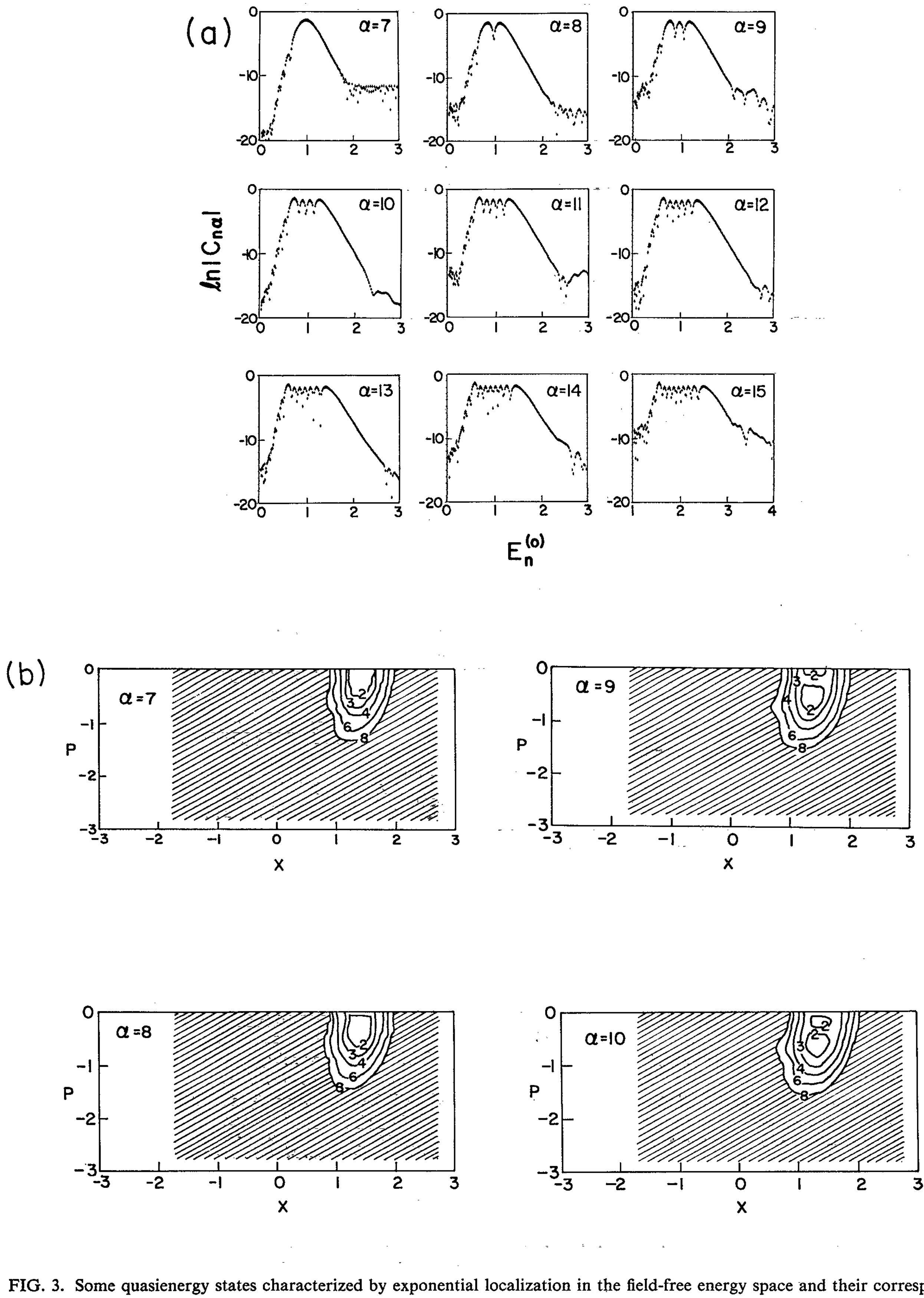

FIG. 3. Some quasienergy states characterized by exponential localization in the field-free energy space and their corresponding Husimi distribution functions show localization in the regular region of classical phase space. The numbers 2,3, ... on the contours should read as $\Gamma(x,p)=10^{-2}$, 10^{-3}, etc. In the hatched region the density is smaller than 10^{-9}.

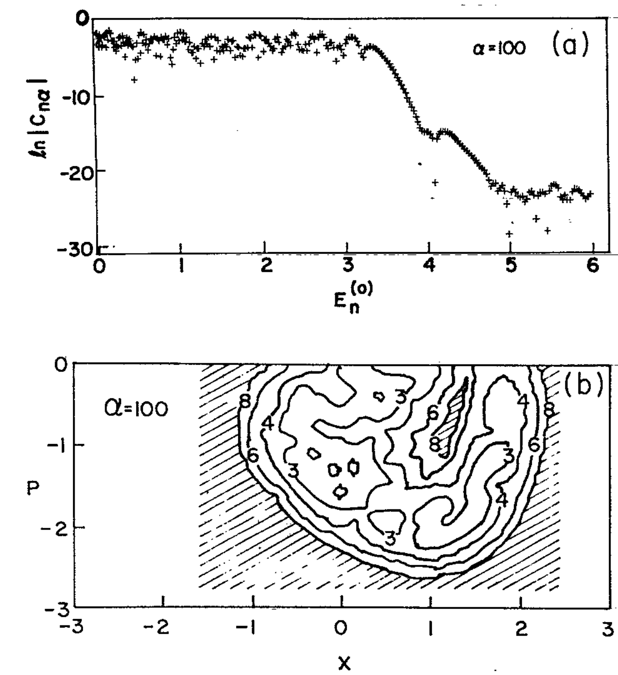

FIG. 4. A characteristic example of an extended quasienergy state expanded in the basis of the field-free states $\psi^{(0)}$. The $C_{n\alpha}=\langle\psi_n^{(0)}|\phi_\alpha\rangle$ were variationally obtained. The Husimi distribution function covers the chaotic region of classical phase space and shows very weak quantum localization. The numbers 2,3,... on the contours should read as $\Gamma(x,p)=10^{-2}$, 10^{-3}, etc. In the hatched region the density is smaller than 10^{-9}.

$\psi_{\alpha=8}\simeq(1/\sqrt{2})(\psi^{(0)}_{73-\Delta_1}-\psi^{(0)}_{73+\Delta_1})$. Similarly, $\psi_{\alpha=9}\simeq(1/\sqrt{3})(\psi^{(0)}_{73-\Delta_2}-\psi^{(0)}_{73}+\psi^{(0)}_{73+\Delta_2})$, etc., where $E_{73\mp n\Delta}^{(0)}\simeq E^{(0)}_{73}\mp n\Delta$, and therefore $\langle\alpha|H_0|\alpha\rangle\simeq E^{(0)}_{73}\approx 1.0$. This explains the significant contribution of $\psi^{(0)}_{73}$ (associated with $E_n^{(0)}$ in Fig. 3) to the odd-α quasienergy states, and the zero contribution to the even-α states. The nodal structure in Fig. 3 is due to the decomposition of the regular quasienergy states as described above.

The other 80 out of the 113 almost-degenerate quasienergy states (see Fig. 2) were *all* found to be extended states, as shown, for example, for $\phi_{\alpha=100}$ in Fig. 4. The results presented there show that the extended quasienergy states having zero (i.e., $<10^{-9}$) probability to be in the regular region of phase space cover almost uniformly all the field-free energies up to $E^{(0)}\leq 4$. Indeed, the classical results presented in Fig. 1 show that for $E^{(0)}>4$ the dynamics is regular. In addition, the regular classical flux tubes in the outer regular region can again be quantized semiclassically by Eq. (9), providing a quantum number of $n\approx 106$ for the lowest of these regular states. This is again in reasonable agreement with the reappearance of regular quasienergy states at $\alpha\approx 113$ (see Fig. 2).

Varying $\hbar$, we found that the number of extended quasienergy states is linearly dependent on the value of $\hbar$. For example, for $\hbar=0.15$ only eight extended states were obtained, and four in a calculation with $\hbar=0.3$.

The Husimi distribution function of *all* the extended states shows very weak localization in the classical phase space (see, for example, the Husimi distribution function presented in Fig. 4), but definitely does *not* show an exponential localization in the field-free energy space, which is so characteristic of the strong quantum limitation of chaos due to the Anderson localization mechanism.

Additional numerical experiments showed that the system studied here is a representative example for other systems for which the chaotic phase space is bounded, such as a periodically driven rigid rotator, which mimics the rotational excitation of diatomic molecules like CsI under the influence of strong electromagnetic fields.

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