

# Suballowable sequences and geometric permutations

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## Abstract

We define *suballowable sequences of permutations* as a generalization of allowable sequences. We give a characterization of allowable sequences in the class of suballowable sequences, prove a Helly-type result on sets of permutations which form suballowable sequences, and show how suballowable sequences are related to problems of geometric realizability. We discuss configurations of points and geometric permutations in the plane. In particular, we find a characterization of pairwise realizability of planar geometric permutations, give two necessary conditions for realizability of planar geometric permutations, and show that these conditions are not sufficient.

*Key words:* Allowable sequence of permutations, suballowable sequence of permutations, configuration of points, geometric permutation.

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## 1 Definitions, notation, background, overview

**U-permutations.** Let  $[n] = \{1, 2, \dots, n\}$ , and let  $p$  be a permutation of the members of  $[n]$ . Denote by  $-p$  the reverse of  $p$ , e.g.,  $(2413) = -(3142)$ . If  $a, b \in [n]$ , and  $a$  occurs before  $b$  in a permutation  $p$ , we denote it by  $p : (\dots a \dots b \dots)$  or by  $p : (a \prec b)$ .

An *undirected permutation* (“*u-permutation*”) is an (unordered) pair formed by a permutation and its reverse:  $\langle 1324 \rangle = \langle 4231 \rangle = \{(1324), (4231)\}$ . A u-permutation will be usually denoted by a lowercase letter with tilde:  $\tilde{p} = \{p, -p\}$ . The permutations that form a u-permutation are its *representatives*.

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U-permutations may be used when one does not need to distinguish a permutation from its reverse.

**Allowable sequences of permutations.** In 1980, Goodman and Pollack [6] introduced the notion of an *allowable sequence of permutations*. It is defined as an infinite sequence of permutations of  $[n]$  satisfying the following:

- (1) It is periodic, and the terms in the second half-period are the reverses of the corresponding terms in the first half-period.
- (2) In the course of a half-period (by the first term of the second half-period), each pair of labels switches exactly once.
- (3) Each term in the sequence is obtained from the previous one by reversing a string, or several disjoint strings, of labels.

The first condition means: There is a natural number  $\ell$  (the length of a half-period of the sequence) such that for each integer  $i$  we have  $p_{i+\ell} = -p_i$ .

The second condition means: If  $p_1 : (a \prec b)$  and  $p_i : (b \prec a)$  for some  $i \in \{2, 3, \dots, \ell\}$ , then  $p_j : (b \prec a)$  for each  $j \in \{i, i+1, \dots, \ell\}$  (note that  $p_{\ell+1} : (b \prec a)$  by the first condition).

An allowable sequence in which each term is obtained from the previous one by reversing exactly one string of the length 2 (that is, by switching one pair of consecutive labels) is called a *simple allowable sequence*.

**Suballowable sequences of permutations.** We define a *suballowable sequence of permutations* as a sequence of permutations of  $[n]$  that satisfies conditions 1 and 2 in the definition of an allowable sequence.

Each suballowable sequence  $\mathcal{L}$  is completely defined by its half-period. Therefore we shall usually write only a half-period of such a sequence, and it will also be called a “suballowable sequence”. For example, the suballowable sequence

$$\dots \rightarrow \underbrace{(1234) \rightarrow (2143) \rightarrow (4213)}_{\text{1st half-period}} \rightarrow \underbrace{(4321) \rightarrow (3412) \rightarrow (3124)}_{\text{2nd half-period}} \rightarrow (1234) \rightarrow \dots$$

may be presented as

$$(1234) \rightarrow (2143) \rightarrow (4213)$$

or, minding the reverse of the first term in the first half-period as the first term in the second half-period, as

$$(1234) \rightarrow (2143) \rightarrow (4213)[\rightarrow (4321)].$$

Note that this representation is not unique: it is possible to begin a half-period at any of the terms.

Denote by  $|\mathcal{L}|$  the length of a half-period of  $\mathcal{L}$ . For any suballowable sequence,  $|\mathcal{L}| \leq \binom{n}{2}$ . Moreover,  $|\mathcal{L}| = \binom{n}{2}$  if and only if  $\mathcal{L}$  is a simple allowable sequence.

**Suballowable (allowable, simple allowable) sets of undirected permutations.** A set of u-permutations of  $[n]$  is a *suballowable (allowable, simple allowable) set* if it is possible to choose a representative of each of its members and to order them into [a half-period of] a suballowable (allowable, simple allowable) sequence.

It is clear that every set  $\mathcal{P}$  of u-permutations with  $|\mathcal{P}| \leq 2$  is suballowable. An example of a non-suballowable set of three u-permutations is  $\mathcal{P} = \{\langle 1234 \rangle, \langle 1342 \rangle, \langle 1423 \rangle\}$ : it can be checked that any sequence of representatives of the members of this set is not suballowable. Below we shall describe a systematic way to check suballowability. We shall also see that a suballowable set of u-permutations can be converted to a suballowable sequence of permutations in an essentially unique way.

The motivation for study of suballowable sequences is the following. Combinatorial properties of allowable sequences of permutations were used in proofs of several geometric results on planar configurations of points, arrangements of pseudolines,  $k$ -sets,  $(\leq k)$ -sets, etc. However, in several cases only the first two conditions in the definition of an allowable sequence were used. For example: Alon and Györi proved [1] that for  $2k < n$ , the maximum number of  $(\leq k)$ -sets of a set of  $n$  points in the plane is  $kn$ . The key argument in their proof was that *in any half-period of an allowable sequence on  $n$  labels, each label participates in at most  $2k$  moves involving the first and the last  $k$  positions*. This is also true for suballowable sequences. (On the other hand, Ungar proved [15] that  $n$  points in the plane (not on the same line) determine at least  $2\lfloor n/2 \rfloor$  directions, using the fact that *this is the minimum length of a non-trivial allowable sequence*. It is clear that this, is general, *does not hold* for suballowable sequences.)

Besides, the combinatorial description of certain geometric settings may involve just suballowable sequences. Below we shall see situations in which using suballowable (rather than allowable) sequences is more natural, or may simplify proofs.

In Section 2 we observe several simple facts on suballowable sequences. In particular, we find a characterization of allowable sequences in the class of suballowable sequences, and show that any suballowable sequence may be completed to an allowable sequence (this means that suballowable sets may be regarded as subsets of allowable sets, hence the name “suballowable”).

In Section 3 we prove a Helly-type combinatorial result on (sub)allowable sets.

In the following two sections, we give two geometric applications of suballowable sequences. Section 4 deals with realizability of sets of permutations by planar configurations of points. Here we demonstrate that non-realizability of an allowable sequence may be determined on its subsequence. Section 5 deals with realizability by planar geometric permutations. We prove that a realizable set must be suballowable. Then we give a characterization of “pairwise realizable” geometric permutations. Thus we have two necessary conditions for realizability: suballowability and pairwise realizability; we show that these two conditions are not sufficient for realizability.

## 2 Basic facts on suballowable sequences

We begin with a few remarks and notation:

- (1) By relabeling, it can be always assumed that the first term in [a half-period of] a suballowable sequence is the natural ordering:  $(123 \dots n)$ .
- (2) Let  $p$  be a permutation. Denote by  $s(p)$  the set of members of  $\binom{[n]}{2}$  switched in  $p$ :  $s(p) = \{\{a, b\} \in \binom{[n]}{2} : a < b, p : (b \prec a)\}$  For example: for  $p = (3412)$ , we have  $s(p) = \{13, 14, 23, 24\}$  (we abbreviate  $\{a, b\}$  by  $ab$ ).
- (3) The corresponding notion for an undirected permutation  $\tilde{p}$  is a 2-partition of  $\binom{[n]}{2}$ : one part contains pairs switched in a representative of  $\tilde{p}$ , the other part contains pairs unswitched in the same representative. This partition will be denoted by  $s(\tilde{p})$ . It is clear that  $s(\tilde{p})$  does not depend on the choice of a representative of  $\tilde{p}$ ; in fact  $s(\tilde{p}) = \{s(p), s(-p)\}$ . Example: for  $\tilde{p} = \langle 3412 \rangle$ , we have  $s(\tilde{p}) = \{\{13, 14, 23, 24\}, \{12, 34\}\}$ . We shall also use the notation  $s(\tilde{p}) = \langle 13, 14, 23, 24 \mid 12, 34 \rangle$ . Sometimes we do not need to mention all the members of  $\binom{[n]}{2}$ ; for example,  $\langle 13, 14 \mid 12, 34 \rangle$  just means that we have a 2-partition in which 13 and 14 belong to one part, and 12 and 34 to another.
- (4) Let  $\mathcal{L}$  be a sequence of permutations of  $[n]$ . A *restriction* of  $\mathcal{L}$  to  $A \subset [n]$  is a sequence of permutations of  $A$  obtained by deleting all the labels from  $[n] \setminus A$  in all the terms of  $\mathcal{L}$  (it may be necessary to delete repeated terms). For  $k \leq n$ , a  $k$ -*restriction* is a restriction of  $\mathcal{L}$  to  $A$  with  $|A| = k$ . An analogous definition applies to sets of permutations.

Suballowable sequences may be described by a wiring diagram in a way similar to that in which allowable sequences are.

It is well known (see, for example, [5]) that an arrangement of vertically monotone pseudolines in the plane “generates” an allowable sequence as follows: scan the arrangement, say, from top to bottom, and record the orders in which

horizontal lines (directed, say, left to right) meet the pseudolines. We show below (Observation 3) that any suballowable sequence may be completed to an allowable sequence. Therefore suballowable sequences also describe arrangements of pseudolines in this way, but the top-to-bottom scanning is discrete and not continuous, and thus not all the permutations are taken in account. Figure 1 presents a realization of an allowable sequence  $(12345) \rightarrow (21543) \rightarrow (25143) \rightarrow (52413) \rightarrow (54231)[\rightarrow (54321)]$  and a “discrete” realization of its subsequence  $(12345) \rightarrow (25143) \rightarrow (54231)[\rightarrow (54321)]$ .

It is easy to see that any suballowable sequence may be described in this way.

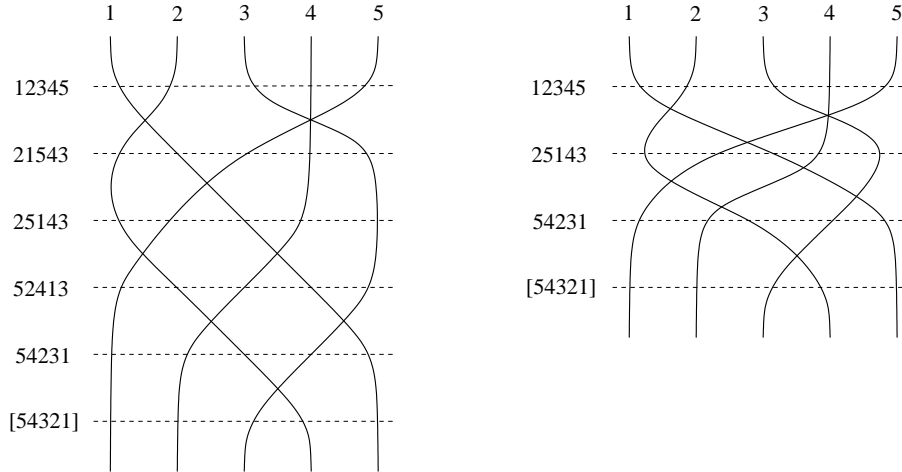


Fig. 1. Wiring diagrams (pseudoline realizations) of an allowable sequence and of its subsequence.

The following proposition is a characterization of allowable sequences among suballowable sequences.

**Lemma 1** *Let  $\mathcal{L}$  be a suballowable sequence. Then  $\mathcal{L}$  is allowable if and only if it does not contain two consecutive terms of the form  $p_i = (\dots a \dots b \dots c \dots)$ ,  $p_{i+1} = (\dots c \dots a \dots b \dots)$  or of the form  $p_i = (\dots a \dots b \dots c \dots)$ ,  $p_{i+1} = (\dots b \dots c \dots a \dots)$ .*

In the other words, a necessary and sufficient condition for a suballowable sequence to be allowable is that its wiring diagram does not contain a move with a restriction of any of the forms presented in Figure 2.

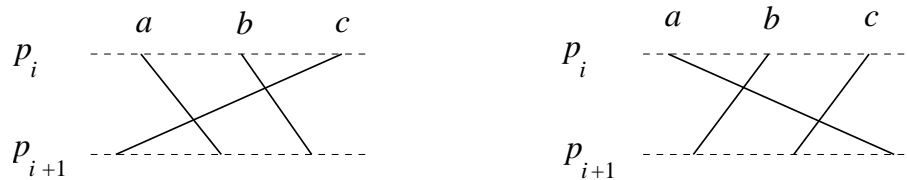


Fig. 2. Two types of moves forbidden in allowable sequences.

**Proof**

$\Rightarrow$  Let  $\mathcal{L}$  be an allowable sequence. Suppose that it contains a move of the form  $(\dots a \dots b \dots c \dots) \rightarrow (\dots c \dots a \dots b \dots)$  (the second case, with  $p_{i+1} = (\dots b \dots c \dots a \dots)$ , is analogous). Since  $a$  and  $c$  switch in this move, these labels belong to the same reversing string. By the same argument,  $b$  and  $c$  belong to the same reversing string. Since each move in an allowable sequence is done by reversing a single string, or several disjoint strings,  $a$  and  $b$  must belong to the same reversing string as well. However, the labels  $a, b$  do not switch in this move, a contradiction.

$\Leftarrow$  We need to prove the following: If a move does not contain a restriction of one of these forms, then it is done by reversing a single string, or several disjoint strings. We assume without loss of generality that the move is  $p \rightarrow q$  with  $p = (1, 2, \dots, n)$ ,  $q = (q_1, q_2, \dots, q_n)$ .

Let  $a$  be the least label such that  $q_a \neq a$ . Denote  $d = q_a$ . Figure 3 (i) shows  $a$  and  $d$  as defined, but we shall see that  $a$  and  $d$  in fact interchange in this move, and thus the situation is as in the Figure 3 (ii). That is,  $a = q_d$ .

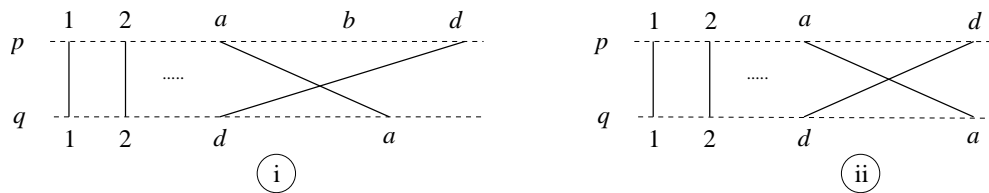


Fig. 3. Illustration to the proof of Lemma 1.

It is clear that  $a$  moves to the right and  $d$  moves to the left in this move. Therefore  $p : (\dots a \dots d \dots)$  and  $q : (\dots d \dots a \dots)$ . We prove that the string  $a \dots d$  in  $p$  and the string  $d \dots a$  in  $q$  (denote these strings by  $p[a \dots d]$  and  $q[d \dots a]$  respectively) contain the same set of labels (in particular, these strings have the same length). For a contradiction, suppose there is a label  $b$  that belongs to the string  $p[a \dots d]$  but does not belong to the string  $q[d \dots a]$ . The label  $b$  cannot occur before  $d$  in  $q$ , therefore it must be after  $a$ . However, then the labels  $a, b, d$  form a forbidden configuration in this move:  $p = (\dots a \dots b \dots d \dots)$ ,  $q = (\dots d \dots a \dots b \dots)$ . The second case, in which there is a label that belongs to  $q[d \dots a]$  but not to  $p[a \dots d]$ , is treated similarly.

Next we show that any pair of labels in the string  $p[a \dots d]$  switches in the move  $p \rightarrow q$ . Suppose  $a \leq b < c \leq d$ , we want to prove that the labels  $b$  and  $c$  switch. If  $c = d$  this is clear: each label in the string (but  $d$  itself) must switch with  $d$ . If  $c < d$  this is also true: if  $b$  and  $c$  do not switch, we have  $p = (\dots b \dots c \dots d \dots)$ ,  $q = (\dots d \dots b \dots c \dots)$ , a contradiction.

We have proved: in the move  $p \rightarrow q$ , the labels  $a$  and  $d$  interchange, and every pair of labels in  $p[a \dots d]$  switches. This means that the whole string  $p[a \dots d]$

reverses in this move.

Now the claim follows inductively. Specifically, we continue “scanning” the labels  $x > d$  in  $p$ . Each time we meet a label that changes place in the move  $p \rightarrow q$ , it results in a reversing string, disjoint from the previous reversing strings. Thus the obtained strings reversing in this move are disjoint. ■

**Observation 2** *Let  $\mathcal{L} = (p_1, p_2, \dots, p_k)$  be a sequence of permutations of  $[n]$  with  $p_1 = (1, 2, \dots, n)$ . Then  $\mathcal{L}$  is [a half-period of] a suballowable sequence if and only if the sequence  $(s(p_1), s(p_2), \dots, s(p_k))$  is a chain with respect to the inclusion of sets.*

This follows immediately from the condition on switching pairs of labels in the definition of suballowable sequence. We shall denote this chain by  $s(\mathcal{L})$ . Note that  $\mathcal{L}$  is [a half-period of] a simple allowable sequence if and only if, in addition, for each  $i \in \{1, 2, \dots, k+1\}$  we have  $|s(p_i)| = i - 1$  (assuming  $p_{k+1} = -p_1$ ).

**Observation 3** *Each suballowable sequence of permutations can be completed to an allowable sequence.*

**Proof** Let  $\mathcal{L}$  be [a half-period of] a suballowable sequence with  $p_1 = (1, 2, \dots, n)$ . We shall show that it is possible to complete  $\mathcal{L}$  to a *simple* allowable sequence.

If  $\mathcal{L}$  is not simple allowable then it contains a move  $p \rightarrow q$  with  $|s(q) \setminus s(p)| > 1$  (including the case when  $p$  is the last term and  $q = (n, n-1, \dots, 1)$ ). There are two labels  $a, b$  such that  $p = (\dots ab \dots)$  and  $q = (\dots b \dots a \dots)$ , i.e., the pair  $ab$  is unswitched *and consecutive* in  $p$ , but it is switched in  $q$ . Such  $a, b$  exist since all the pairs switched in  $p$  remain switched in  $q$ , and if, in addition, all unswitched consecutive pairs remain unswitched in  $q$ , then  $p = q$ .

Let  $p'$  be the permutation obtained from  $p$  by switching the pair  $ab$ . We have  $p' \neq q$  because we assume that  $ab$  was not the only pair that switches in the move  $p \rightarrow q$ . Thus  $s(p) \subset s(p') \subset s(q)$ , and therefore  $\mathcal{L}$  with  $p'$  inserted between  $p$  and  $q$  is again a suballowable sequence which contains one term more than the original one.

We continue to modify the sequence in this way until we get a sequence with  $\binom{n}{2}$  terms. At this stage we have a simple allowable sequence. ■

**Observation 4** *Any restriction of a suballowable (allowable, simple allowable) sequence is a suballowable (allowable, simple allowable) sequence.*

**Proof**

*For suballowable sequences:* Let  $\mathcal{L}$  be [a half-period of] a suballowable sequence with  $p_1 = (1, 2, \dots, n)$ . Consider the chain  $s(\mathcal{L})$ . If  $\mathcal{L}'$  is a restriction of  $\mathcal{L}$  to  $A \subset [n]$ , then  $s(\mathcal{L}')$  is obtained from  $s(\mathcal{L})$  by deleting all the terms that contain labels from  $[n] \setminus A$ . It is clear that  $s(\mathcal{L}')$  is also a chain, therefore  $\mathcal{L}'$  is [a half-period of] a suballowable sequence.

*For allowable sequences:* Let  $\mathcal{L}$  be [a half-period of] an allowable sequence, and let  $\mathcal{L}'$  be restriction of  $\mathcal{L}$ . By the previous case,  $\mathcal{L}'$  is suballowable. If  $\mathcal{L}'$  is not allowable, then it contains a “forbidden” move of a form mentioned in Lemma 1. Then the corresponding move in  $\mathcal{L}$  is also of this form, a contradiction.

*For simple allowable sequences:* Let  $\mathcal{L}$  be [a half-period of] a simple allowable sequence with  $p_1 = (1, 2, \dots, n)$ , and let  $\mathcal{L}'$  be restriction of  $\mathcal{L}$ . By the previous case,  $\mathcal{L}'$  is allowable. If it is not simple allowable, then it contains a move  $p \rightarrow q$  with  $|s(q) \setminus s(p)| > 1$ . Then the same happens in the corresponding move in  $\mathcal{L}$ , a contradiction.  $\blacksquare$

### 3 A Helly-type result on (sub)allowable sets

We prove a Helly-type result on labels of suballowable *sets* of u-permutations.

**Theorem 5** *Let  $\mathcal{P}$  be a set of undirected permutations of  $[n]$ , with  $n \geq 6$ .  $\mathcal{P}$  is suballowable if and only if all the 6-restrictions of  $\mathcal{P}$  are suballowable.*

#### Proof

$\Rightarrow$  This follows immediately from Observation 4: one just needs to convert  $\mathcal{P}$  to a half-period of a suballowable sequence and to take the restriction.

$\Leftarrow$  Assume, without loss of generality, that  $\mathcal{P}$  contains the natural order u-permutation  $\tilde{p}_1 = \langle 1, 2, \dots, n \rangle$ .

By Observation 2,  $\mathcal{P} = \{\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k\}$  is a suballowable set of u-permutations if and only if  $s(\mathcal{P})$  forms a chain, i.e., if it is possible to choose a representative of each of the 2-partitions  $s(\tilde{p}_1), s(\tilde{p}_2), \dots, s(\tilde{p}_k)$  so that the chosen sets form a chain.

We prove a lemma characterizing those families of 2-partitions which form a chain. A clear necessary condition is that every two partitions are in an inclusion relation:

**Definition** Let  $X$  be a set. For  $A \subseteq X$ , denote  $\bar{A} = X \setminus A$ . Let  $\mathcal{A} = \langle A | \bar{A} \rangle$ ,  $\mathcal{B} = \langle B | \bar{B} \rangle$  be two 2-partitions of  $X$ . We say that  $\mathcal{A}$  and  $\mathcal{B}$  are *comparable* if

there is an inclusion between a pair of their representatives. Such an inclusion is always accompanied by the reverse inclusion of the complements, and thus there are four possibilities for comparability of  $\mathcal{A}$  and  $\mathcal{B}$ : (1)  $A \subseteq B, \bar{A} \supseteq \bar{B}$ ; (2)  $A \subseteq \bar{B}, \bar{A} \supseteq B$ ; (3)  $\bar{A} \subseteq B, A \supseteq \bar{B}$ ; (4)  $\bar{A} \subseteq \bar{B}, A \supseteq B$ .

For families of *sets*, if every two sets are comparable, then the family forms a chain. For families of 2-partitions this is not true, as the following example shows: Let  $X = \{a, b, c\}$ , and consider the following partitions:  $\mathcal{A} = \langle a | b, c \rangle$ ,  $\mathcal{B} = \langle b | a, c \rangle$ ,  $\mathcal{C} = \langle c | a, b \rangle$ . Every two of them are comparable, but they do not form a chain. However, we shall see that only this situation may prevent from pairwise comparable family of 2-partitions to form a chain.

**Definition** Let  $X$  be a set, and let  $\mathcal{F}$  be a family of 2-partitions of  $X$ . We say that three members  $a_0, a_1, a_2$  of  $X$  are *separated* by  $\mathcal{F}$  if for each  $i \in \{0, 1, 2\}$  there is a partition  $\mathcal{A}_i \in \mathcal{F}$  of the form  $\langle a_i | a_{i+1}, a_{i+2} \rangle$  (the addition in indices is mod 3).

**Lemma 6** *Let  $X$  be a set, and let  $\mathcal{F} = \{\mathcal{A}_i = \langle A_i | \bar{A}_i \rangle\}_{i \in [k]}$  be a set of 2-partitions of  $X$ . Suppose that every two partitions in  $\mathcal{F}$  are comparable, and that no three members of  $X$  are separated by  $\mathcal{F}$ . Then  $\mathcal{F}$  forms a chain.*

**Proof of Lemma 6** It can be assumed that all the members of  $\mathcal{F}$  are distinct, and that  $\mathcal{F}$  does not contain the trivial partition  $\langle X | \emptyset \rangle$ .

We prove the Lemma by an inductive construction of the chain. At  $i$ -th step we add a representative of  $\mathcal{A}_i$  to the chain.

The first step: Choose any representative of  $\mathcal{A}_1$ . Since the representatives are not distinguished, we can assume without loss of generality that we choose  $A_1$ .

The second step: Since every two partitions in  $\mathcal{F}$  are comparable, either  $A_2$  or  $\bar{A}_2$  includes  $A_1$  or is included in it. It is possible to assume without loss of generality that it is  $A_2$ ; since we can relabel the members of  $\mathcal{F}$ , we also assume that  $A_1 \subset A_2$ .

The general step: Let  $2 \leq m < k$ . By now, we have a chain of representatives of  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$ . By relabeling, we assume that  $A_1 \subset A_2 \subset \dots \subset A_m$ . We need to add a representative of the partition  $\mathcal{A}_{m+1}$  to this chain. By the assumption, each one of the sets  $A_1, A_2, \dots, A_m$  is comparable with one of the sets  $A_{m+1}, \bar{A}_{m+1}$ . Without loss of generality,  $A_1$  is comparable with  $A_{m+1}$ . If  $A_{m+1} \subset A_1$ , then  $A_{m+1} \subset A_1 \subset A_2 \subset \dots \subset A_m$  is a chain. Suppose  $A_1 \subset A_{m+1}$ . Let  $x$  be the maximal  $i \in \{1, 2, \dots, m\}$  such that  $A_x \subset A_{m+1}$ . If  $x = m$ , we have the chain  $A_1 \subset A_2 \subset \dots \subset A_m \subset A_{m+1}$ . Otherwise,  $A_1 \subset A_2 \subset \dots \subset A_x \subset A_{m+1}$ , but  $A_{x+1} \not\subset A_{m+1}$ . We want to show:  $A_{m+1} \subset A_{x+1}$ . Indeed, if  $A_{x+1} \not\subset A_{m+1}$  and  $A_{m+1} \not\subset A_{x+1}$  then the sets  $A_{x+1} \setminus A_{m+1}$  and

$A_{m+1} \setminus A_{x+1}$  are not empty; on the other hand  $A_{x+1} \cap A_{m+1}$  is not empty (it includes  $A_1$  which is nonempty). Let  $a \in A_1$ ,  $b \in A_{x+1} \setminus A_{m+1}$ ,  $c \in A_{m+1} \setminus A_{x+1}$ . We have  $a \in A_1$ ,  $a \in A_{x+1}$ ,  $a \in A_{m+1}$ ;  $b \notin A_1$ ,  $b \in A_{x+1}$ ,  $b \notin A_{m+1}$ ;  $c \notin A_1$ ,  $c \notin A_{x+1}$ ,  $c \in A_{m+1}$ . This means that the partitions  $\mathcal{A}_1$ ,  $\mathcal{A}_{x+1}$ ,  $\mathcal{A}_{m+1}$  separate  $a$ ,  $b$  and  $c$ :  $\mathcal{A}_1 : \langle a | b, c \rangle$ ,  $\mathcal{A}_{x+1} : \langle c | a, b \rangle$ ,  $\mathcal{A}_{m+1} : \langle b | a, c \rangle$ , a contradiction.

Thus  $A_{m+1} \subset A_{x+1}$ , and we obtain the chain  $A_1 \subset A_2 \subset \cdots \subset A_x \subset A_{m+1} \subset A_{x+1} \subset \cdots \subset A_m$ . For the next step we relabel the partitions (and the sets in the chain) and thus we have finally  $A_1 \subset A_2 \subset \cdots \subset A_m \subset A_{m+1}$ . ■

In our context,  $X = \binom{[n]}{2}$ . Note the following: at the first step one chooses an arbitrary representative of  $\mathcal{A}_1$ , after that the construction of the chain is completed in a unique way. This means that if  $\mathcal{P}$  is a suballowable set of  $k$  u-permutations ( $k \geq 2$ ), then it can be converted into a suballowable sequence in precisely two ways which are the reverses of each other. It follows that there are exactly  $4k$  possibilities to write a half-period of a suballowable sequence whose terms belong to  $\mathcal{P}$ : Note that if  $(p_1 \rightarrow p_2 \rightarrow \cdots \rightarrow p_{k-1} \rightarrow p_k)$  is a half-period of a suballowable sequence, then  $(p_2 \rightarrow p_3 \rightarrow \cdots \rightarrow p_k \rightarrow -p_1)$  is a half-period of the same suballowable sequence. This gives  $2k$ ; another  $2k$  ways are obtained by reversing the sequence:  $(p_k \rightarrow p_{k-1} \rightarrow \cdots \rightarrow p_2 \rightarrow p_1)$ ,  $(p_{k-1} \rightarrow p_{k-2} \rightarrow \cdots \rightarrow p_1 \rightarrow -p_k)$  and so on.

The proof of Lemma 6 also suggests a way of checking whether a set  $\mathcal{P}$  of u-permutations is suballowable: choose  $(1, 2, \dots, n)$  as a first term, any representative of any other u-permutation as a second term, and then, using the sets  $s(p_i)$  of switched pairs, insert representatives of other u-permutations by one. In this way one either constructs a suballowable sequence or fails to insert a term, identifying an explicit obstruction to suballowability.

We return to the proof of Theorem 5. Let  $\mathcal{P}$  be a non-suballowable set of u-permutations. This means: the family  $s(\mathcal{P})$  of partitions of  $\binom{[n]}{2}$  does not form a chain.

By Lemma 6, either three elements  $T, U, V$  of  $\binom{[n]}{2}$  are separated by  $s(\mathcal{P})$ , or there are two non-comparable partitions in  $s(\mathcal{P})$ .

If the first case, there are three partitions in  $s(\mathcal{P})$  with  $\langle T | U, V \rangle$ ,  $\langle U | T, V \rangle$ ,  $\langle V | T, U \rangle$ . If we take the restriction of  $\mathcal{P}$  to  $T \cup U \cup V$ , then in the restricted family of partitions  $T, U, V$  are still separated, therefore this restriction is also non-suballowable. Since  $|T \cup U \cup V| \leq 6$ , we have a non-suballowable restriction of  $s(\mathcal{P})$  on at most 6 labels.

If the second case, there are four elements  $T, U, V, W$  of  $\binom{[n]}{2}$  and two partitions  $\mathcal{A}, \mathcal{B}$  in  $s(\mathcal{P})$  of the form

$$\mathcal{A} = \langle T, U \mid V, W \rangle,$$

$$\mathcal{B} = \langle T, V \mid U, W \rangle.$$

Denote  $\mathcal{T} = \{T, U, V, W\}$ . Take the restriction of  $\mathcal{P}$  to  $\cup \mathcal{T}$ . Since  $|\cup \mathcal{T}| \leq 8$ , we have a non-suballowable restriction of  $s(\mathcal{P})$  on at most 8 labels. Now we shall show how to reduce 8 to 6. We shall see that if  $|\cup \mathcal{T}| > 6$  then it is possible to modify  $\mathcal{T}$  so that the members of the modified  $\mathcal{T}$  will contain all together at most six labels.

Suppose  $|\cup \mathcal{T}|$  is 7 or 8. Then there are two members of  $\mathcal{T}$  disjoint from each other and from all other members of  $\mathcal{T}$  – denote them by  $T_1$  and  $T_2$ . Consider  $Z \in \binom{[n]}{2}$  such that  $|Z \cap T_1| = |Z \cap T_2| = 1$ . One of the members of  $\mathcal{T}$  belongs to the same parts of  $\mathcal{A}$  and  $\mathcal{B}$  as  $Z$ ; assume without loss of generality that it is  $T$ . Let  $\mathcal{T}' = (\mathcal{T} \setminus \{T\}) \cup \{Z\}$ . It is clear that the members of  $\mathcal{T}'$  are an obstruction to comparability as the members of  $\mathcal{T}$  were; in addition  $|\cup \mathcal{T}'| < |\cup \mathcal{T}|$  since there is  $a \in T \setminus (U \cup V \cup W \cup Z)$ : this  $a$  does not belong to  $\cup \mathcal{T}'$ .

Performing this procedure once or twice, we get the modified  $\mathcal{T}$  with  $|\cup \mathcal{T}| \leq 6$ . Therefore there is a non-suballowable (at most)-6-restriction of  $\mathcal{P}$ .  $\blacksquare$

**Remark** The Helly number 6 is the best possible as the following example shows:  $\mathcal{P} = \{\tilde{p}_1 = \langle 123456 \rangle, \tilde{p}_2 = \langle 124365 \rangle, \tilde{p}_3 = \langle 213465 \rangle\}$ . The corresponding partitions are  $s(\tilde{p}_1) = \{\emptyset, \binom{[6]}{2}\}$ ,  $s(\tilde{p}_2) = \{\{34, 56\}, \binom{[6]}{2} \setminus \{34, 56\}\}$ ,  $s(\tilde{p}_3) = \{\{12, 56\}, \binom{[6]}{2} \setminus \{12, 56\}\}$ . They do not form a chain, thus  $\mathcal{P}$  is non-suballowable.

However, all the 5-restrictions of  $\mathcal{P}$  are suballowable:

The restriction to  $[6] \setminus \{2\}$  is suballowable since the sequence  $(13456) \rightarrow (13465) \rightarrow (14365)$  is suballowable;

The restriction to  $[6] \setminus \{4\}$  is suballowable since the sequence  $(12356) \rightarrow (12365) \rightarrow (21365)$  is suballowable;

The restriction to  $[6] \setminus \{6\}$  is suballowable since the sequence  $(12345) \rightarrow (21345) \rightarrow (53421)$  is suballowable;

The cases of  $[6] \setminus \{1\}$ ,  $[6] \setminus \{3\}$ ,  $[6] \setminus \{5\}$  are similar.

We deduce that the same result holds also for allowable and for simple allowable sets:

**Corollary 7** *Let  $\mathcal{P}$  be a set of  $u$ -permutations of  $[n]$ , with  $n \geq 6$ .  $\mathcal{P}$  is allowable (simple allowable) if and only if all the 6-restrictions of  $\mathcal{P}$  are allowable*

(*simple allowable*).

### Proof

$\Rightarrow$  Follows directly from Observation 4.

$\Leftarrow$  *for allowable sets*: Since all the 6-restrictions of  $\mathcal{P}$  are allowable, in particular they are suballowable. By Theorem 5,  $\mathcal{P}$  is suballowable. Suppose it is not allowable. Then it contains a “forbidden” move of a form mentioned in Lemma 1. Take any restriction containing three labels which form a “forbidden” configuration. This restriction is also not allowable, a contradiction.

$\Leftarrow$  *for simple allowable sets*: Since all the 6-restrictions of  $\mathcal{P}$  are simple allowable, in particular they are allowable. By the previous case,  $\mathcal{P}$  is allowable. Suppose it is not simple allowable.

Convert  $\mathcal{P}$  into an allowable sequence (we saw that this can be done in an essentially unique way) and assume that  $p_1 = (1, 2, \dots, n)$ . Since the sequence is not simple allowable, it contains a move  $p \rightarrow q$  such that  $|s(q) \setminus s(p)| > 1$ . This means: at least two pairs of labels switch in this move. Let  $ab$  and  $cd$  be two such pairs (not necessarily disjoint). Consider the restriction of the sequence to  $\{a, b, c, d\}$ . In the corresponding chain  $|s(q) \setminus s(p)| > 1$  holds, therefore this restriction is not simple allowable, a contradiction. ■

**Remark** In contrast to what happens in Theorem 5, we do not know whether the number 6 in Corollary 7 is the best possible. Probably it may be replaced by 4 (but not by 3: take  $\mathcal{P}$  to be the set of all u-permutations of  $[n]$ ).

## 4 Configurations of points

The motivation for introducing allowable sequences in [6] was the following observation: Let  $\mathcal{F}$  be a set of  $n$  points in the plane, labeled by  $[n]$ . Take a directed line  $\ell$  and note the permutation of  $[n]$  which describes the order in which the projections of the members of  $\mathcal{F}$  appear on this line. Rotate  $\ell$  (say, counterclockwise) and write the sequence of all the permutations that appear in this way (ignoring the finite number of directions of  $\ell$  when it is perpendicular to a line defined by a pair of points from  $\mathcal{F}$  and the permutation is not defined). This sequence is called *the circular sequence of  $\mathcal{F}$* . It is easy to see that this sequence is allowable. Thus allowability is a necessary condition on a sequence of permutation to encode in this way a configuration of points in  $\mathbb{R}^2$ . However, as several examples show, it is not sufficient; in other words, not every allowable sequence is actually realizable by a planar configuration of points. We observe that if an allowable sequence is non-realizable in this

sense, then its non-realizability may be determined on its subsequence. Therefore it makes sense to speak about *realizable and non-realizable suballowable* sequences.

Formally, a suballowable sequence  $\mathcal{L}$  of permutations of  $[n]$  is realizable by a planar configuration of points if there exists a finite set  $\mathcal{F}$  of  $n$  points in the plane, labeled by  $[n]$ , such that  $\mathcal{L}$  is a subsequence of the circular sequence of  $\mathcal{F}$ . A suballowable *set* of u-permutations is realizable if there is a way to choose representative of each of its members and convert them into a suballowable sequence.

Since suballowable sequences have, in general, less terms than allowable sequences, proofs of non-realizability may be easier for suballowable sequences. Proposition 8 presents an example of non-realizable suballowable sequence. We prove it using the non-stretchability of certain pseudoline arrangement sequence; however it can be also proved directly, using Pappus' theorem (a pseudoline arrangement is *stretchable* if there exists an isomorphic arrangement of straight lines).

**Proposition 8** *The suballowable sequence  $\mathcal{L}$  with half-period  $(123456) \rightarrow (412563) \rightarrow (541632)[\rightarrow (654321)]$  is not realizable by a planar configuration of points.*

**Proof** In order to show that  $\mathcal{L}$  is non-realizable, we use the fact that an allowable sequence is realizable by a configuration of points if and only if it is realizable by an arrangement of straight lines [6] (it is clear that this is true also for suballowable sequences). Therefore, if  $\mathcal{L}$  is realizable then there exist six non-horizontal lines  $t_1, \dots, t_6$  (to be abbreviated just by  $1, \dots, 6$ ), and three horizontal lines  $a, b, c$  ( $b$  between  $a$  and  $c$ ) directed left to right, so that  $a$  meets the lines  $t_i$  in the order  $(1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 6)$ ,  $b$  meets them in the order  $(4 \prec 1 \prec 2 \prec 5 \prec 6 \prec 3)$ , and  $c$  meets them in the order  $(5 \prec 4 \prec 1 \prec 6 \prec 3 \prec 2)$ , see Figure 4. We have a part of a pseudoline arrangement. It can be completed to an arrangement in an essentially unique way: if we extend the pseudolines below, 1 meets 6, 3, 2 just in this order. Similarly, 6 meets 1, 4, 5 just in this order. There are two ways in which  $a, b$  and  $c$  can intersect each other; however, they give isomorphic arrangements. Thus we obtain an arrangement depicted in Figure 5 (i). It turns out that it is isomorphic to the well-known non-stretchable arrangement due to Ringel [10] – it appears in Figure 5 (ii).

## Remarks

- (1) In fact we have a stronger result: that the (suballowable) *set* of u-permutations  $\{\langle 123456 \rangle, \langle 412563 \rangle, \langle 541632 \rangle\}$  is not realizable. This follows from the fact that a suballowable set may be converted into a suballowable sequence in

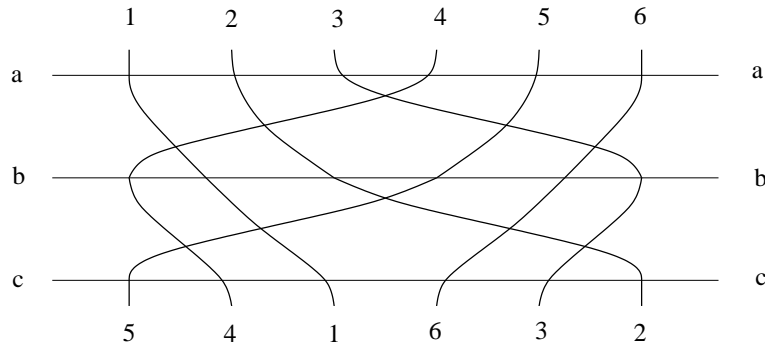


Fig. 4. A pseudoline realization of  $(123456) \rightarrow (412563) \rightarrow (541632)$ .

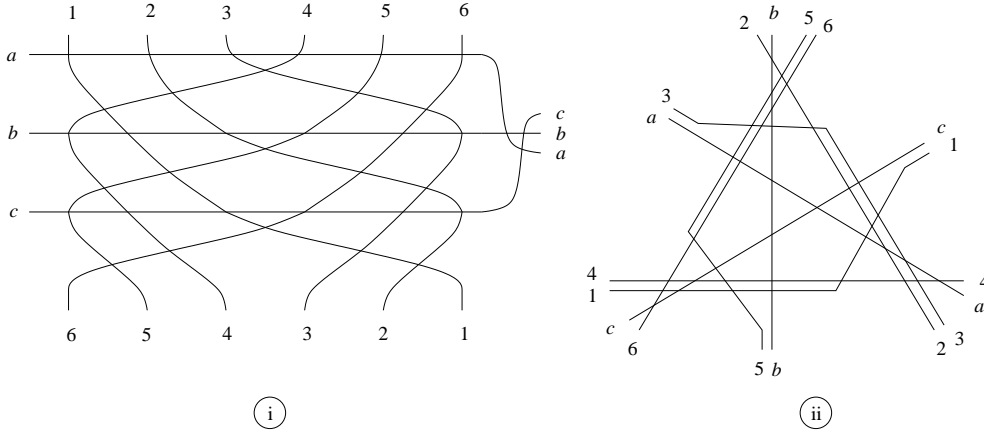


Fig. 5. Illustration to the proof of Proposition 8.

- an essentially unique way (see the remark after the proof of Lemma 6).
- (2) Note that  $\mathcal{L}$  can be completed to an allowable sequence in several ways. Therefore it gives at once several examples of non-realizable allowable sequences.
  - (3) The sequence  $\mathcal{L}$  is “minimal” non-realizable in two senses: (1) If we take a restriction of  $\mathcal{L}$  to any five labels, we obtain a realizable suballowable sequence. (2) If we delete a term in  $\mathcal{L}$  then we obtain a suballowable sequence with two terms which is obviously realizable.
  - (4) We have constructed examples which show that  $\mathcal{L}$  is the only, up to relabeling, non-realizable suballowable sequence on six labels with three terms.
  - (5) Given an allowable sequence, the problem of determining whether it is realizable by a planar configuration of points (or, equivalently, by an arrangement of straight lines) is NP-hard [11]. May it happen that the analogous problem for suballowable sequences with three terms is easier? In any case, we expect that the situation is similar to that with simple pseudoline arrangements, where stretchability cannot be guaranteed by the exclusion of a finite number of forbidden subarrangements (see [3,

## 5 Geometric permutations

A *geometric permutation* of a finite family  $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$  of pairwise disjoint convex sets in  $\mathbb{R}^d$  is the order in which an undirected transversal line meets the members of  $\mathcal{F}$ , described by a u-permutation of  $[n]$ . The notion of geometric permutation was introduced by Katchalski, Lewis, Liu and Zaks [8,9]. It was used in proofs of several geometric Helly-type results; a notable example is Tverberg’s proof of Grünbaum’s conjecture: *A planar family of pairwise disjoint translates of a fixed convex set has a transversal line if each subfamily of size 5 does* [14]. For geometric permutations, the problem of realizability arises as well: for fixed  $d$ , given a set  $\mathcal{P}$  of u-permutations, is there a family  $\mathcal{F}$  of pairwise disjoint convex sets in  $\mathbb{R}^d$  such that all the members of  $\mathcal{P}$  are geometric permutations of  $\mathcal{F}$ ? We shall consider this problem in the plane. Note that directed transversal lines may be considered as well: then geometric permutations are permutations rather than u-permutations; this is not essential for realizability problems. *In the remainder of the paper, “realizable set of (u-)permutations” means “set of (u-)permutations realizable as a set of geometric permutations of a family of pairwise disjoint convex sets in  $\mathbb{R}^2$ ”; and “family of disjoint sets” means “family of pairwise disjoint sets”.*

### 5.1 Geometric permutations and suballowable sequences

The following result shows how suballowable sequences are related to realizability of sets of u-permutations by planar geometric permutations.

**Proposition 9** *Let  $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$  be a family of disjoint convex sets in  $\mathbb{R}^2$ , and let  $\mathcal{P}$  be the set of geometric permutations of  $\mathcal{F}$ . Then  $\mathcal{P}$  is a suballowable set.*

**Proof** For each geometric permutation in  $\mathcal{P}$ , choose one transversal line that induces it. Direct the chosen transversal lines so that their directions will be in the range  $[0, \pi)$  (the upper semicircle  $\Sigma$  of the circle of directions). Convert  $\mathcal{P}$  into a sequence by choosing the representatives that correspond to these directions, and ordering them by their directions. We shall show that this sequence is [a half-period of] a suballowable sequence.

Let  $a, b \in [n]$ . Consider a line  $m$  that separates  $A_a$  from  $A_b$ , and translate it to the circle of directions. This line divides  $\Sigma$  into two sectors (see Figure 6). Then  $a \prec b$  on the transversals that belong to one of them, and  $b \prec a$  on the

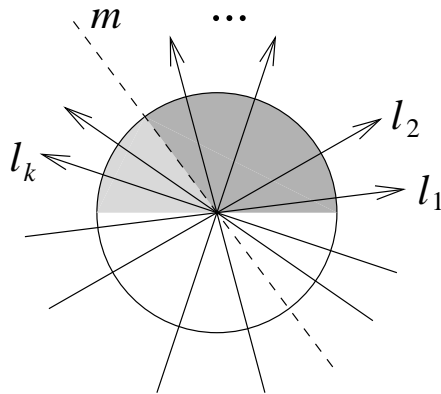


Fig. 6. Illustration to the proof of Proposition 9.

transversals that belong to another.

Thus, in the sequence constructed in this way from representatives of the members of  $\mathcal{P}$ , every pair of labels switches just once, and thus it is [a half-period of] a suballowable sequence. ■

Thus, suballowability is a necessary condition on a set of  $u$ -permutations for realizability by planar geometric permutations. Therefore it makes sense to speak about realizable and non-realizable *suballowable sequences* of permutations (rather than sets of  $u$ -permutations). However, suballowability is not sufficient for realizability. In particular, this follows from a theorem by Edelsbrunner and Sharir [4] which asserts that a planar family of  $n$  disjoint convex sets admits at most  $2n - 2$  geometric permutations, whereas a suballowable sequence may contain up to  $\binom{n}{2}$  terms.

## 5.2 Pairwise realizability of planar geometric permutations

In general, the problem of deciding whether a given set of permutations is realizable by planar geometric permutations is expected to be a difficult one. However, it turns out that for sets of *two* permutations realizability can be characterized easily.

We shall use the following notation and definition.

Let  $p$  be a permutation of  $[n] = \{1, 2, \dots, n\}$ . Let  $A \subseteq [n]$ . We denote by  $p|_A$  the restriction of  $p$  on  $A$ . For example, for  $n = 5$ ,  $p = (25413)$  and  $A = \{1, 3, 5\}$ , we have  $p|_A = (513)$ .

**Definition** A permutation  $p$  is *decomposable* if there exists a partition  $\{K, M\}$  of  $[n]$  ( $K \cup M = [n]$ ,  $K \cap M = \emptyset$ ) such that  $p|_K$  is ascending, and  $p|_M$  is

descending.

Note that a permutation is decomposable if and only if its reverse is decomposable ( $K$  and  $M$  interchange roles), and thus the property of being decomposable is well defined for u-permutations.

An example of a non-decomposable permutation is (2143). It was noted in one of the first papers on geometric permutations [8] that the pair  $\{\langle 1234 \rangle, \langle 2143 \rangle\}$  is non-realizable in  $\mathbb{R}^2$ . The next theorem generalizes this observation.

**Theorem 10** *Let  $\{p = (1, 2, \dots, n), q = (b_1, b_2, \dots, b_n)\}$  be a set of two permutations of  $[n]$ . The following statements are equivalent:*

- (1)  $\{p, q\}$  is realizable in  $\mathbb{R}^2$ ;
- (2)  $q$  is decomposable;
- (3) There are no  $x, y, z, t \in [n]$ ,  $x < y < z < t$  such that  $q = \pm(\dots y \dots x \dots t \dots z \dots)$ .
- (4) Each 4-restriction of  $\{p, q\}$  is realizable.

Here we assume that  $p$  is the natural-ordering permutation. We shall prove the result under this assumption; it is clear that the following, more general, statement follows immediately by relabeling:

**Theorem 11** *Let  $\{p = (a_1, a_2, \dots, a_n), q = (b_1, b_2, \dots, b_n)\}$  be a set of two permutations of  $[n]$ . The following statements are equivalent:*

- (1)  $\{p, q\}$  is realizable in  $\mathbb{R}^2$ ;
- (2)  $[n]$  can be partitioned to two subsets ( $K \cup M = [n]$ ,  $K \cap M = \emptyset$ ) so that the members of  $K$  occur in  $p$  and  $q$  in the same order, and the members of  $M$  occur in  $p$  and  $q$  in the opposite orders ( $p|_K = q|_K$  and  $p|_M = -(q|_M)$ );
- (3) There are no  $x, y, z, t \in [n]$  such that  $p = (\dots x \dots y \dots z \dots t \dots)$ ,  $q = \pm(\dots y \dots x \dots t \dots z \dots)$ .
- (4) Each 4-restriction of  $\{p, q\}$  is realizable.

**Remark** Note that suballowability is not involved here, as every sequence with two terms is suballowable.

### Proof of $1 \Leftrightarrow 2$ in Theorem 10

$\Rightarrow$  Let  $\{p = (1, 2, \dots, n), q = (b_1, b_2, \dots, b_n)\}$  be a realizable set of permutations of  $[n]$ . Consider a family  $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$  that realizes it with directed transversal lines  $l_p, l_q$  which induce permutations  $p, q$  respectively. Then  $\{p, q\}$  can be also realized by a family of disjoint *segments* with endpoints on  $l_p, l_q$ : for  $i = 1, \dots, n$ , choose a pair of points  $P_i \in A_i \cap l_p$ ,  $Q_i \in A_i \cap l_q$ , and replace  $A_i$  by  $\text{conv}(P_i, Q_i)$ . Using standard arguments we can also avoid the situation in which one of the segments contains  $O$ , the point

of intersection of  $l_p$  and  $l_q$ . Thus we shall assume that  $\mathcal{F}$  is such a realization. Each  $A_i \in \mathcal{F}$  belongs to one of quadrants denoted as in Figure 7.

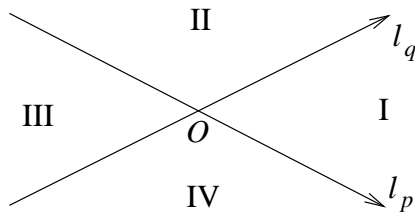


Fig. 7. Four quadrants defined by  $l_p$  and  $l_q$ .

Denote  $K = \{i \in [n] : A_i \in \text{I} \cup \text{III}\}$ ,  $M = \{i \in [n] : A_i \in \text{II} \cup \text{IV}\}$ . Then  $K \cup M = [n]$ ,  $K \cap M = \emptyset$ . It is easy to see that  $q|_K$  is ascending, and  $q|_M$  is descending, see Figure 8. Thus  $q$  is decomposable.

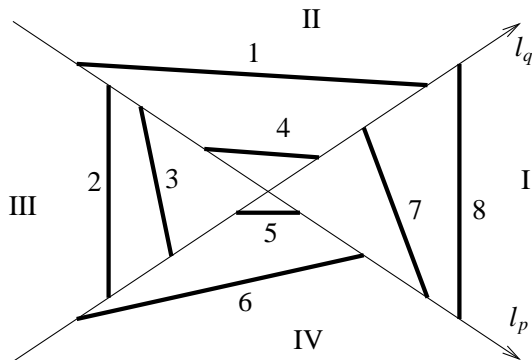


Fig. 8. A realization of  $\{p = (12345678), q = (62354718)\}$ : here  $K = \{2, 3, 7, 8\}$ ,  $M = \{1, 4, 5, 6\}$ , and  $q|_K = (2378)$ ,  $q|_M = (6541)$ .

$\Leftarrow$  Consider the graph of the permutation  $q = (b_1, b_2, \dots, b_n)$  (it is the point set  $\{(i, b_i) : i \in [n]\}$ ). Atkinson [2] proved that the graph of a decomposable permutation  $q$  has the form shown in Figure 9 (i), where the points of  $A$  and of  $E$  are ascending, the points of  $B$  and of  $D$  are descending, and the points of  $C$  are either ascending or descending. Figure 9 (ii) demonstrates this for the permutation  $(827354619)$ : here  $A = \{2, 3\}$ ,  $B = \{7, 8\}$ ,  $C = \{4, 5\}$ ,  $D = \{1\}$ ,  $E = \{6, 9\}$ .

It is possible to construct a realization of  $\{p, q\}$  as follows. Take a pair of intersecting directed lines,  $l_p$  and  $l_q$ . Let  $O$  denote the point of their intersection. Place points  $P_1, \dots, P_n$  on  $l_p$  so that on  $l_p$

- $P_1 \prec \dots \prec P_n$ ,
- for each  $j \in A \cup D \cup C$ , we have  $P_j \prec O$ ,
- for each  $j \in B \cup E$ , we have  $O \prec P_j$ .

Place points  $Q_1, \dots, Q_n$  on  $l_q$  so that on  $l_q$

- $Q_{b_1} \prec \dots \prec Q_{b_n}$ ,

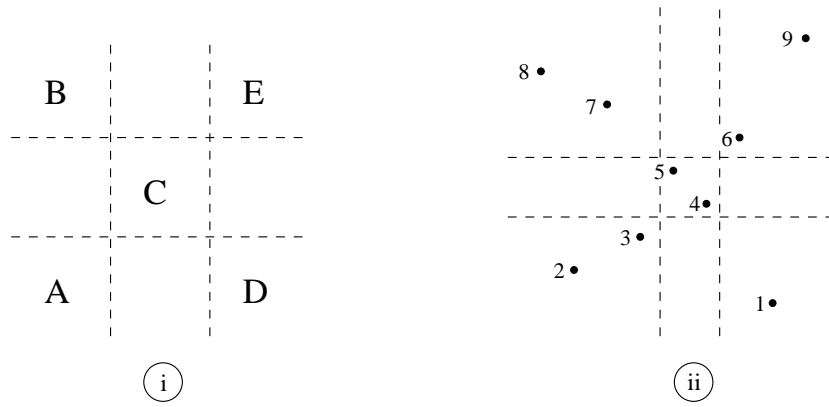


Fig. 9. The graph of a decomposable sequence: in general and for the permutation (827354619).

- for each  $j \in A \cup B$ , we have  $Q_j \prec O$ ,
- for each  $j \in D \cup E$ , we have  $O \prec Q_j$ ,
- if  $C$  is ascending (resp. descending), for each  $j \in C$ , we have  $Q_j \prec O$  (resp.  $O \prec Q_j$ ).

This is possible in view of Atkinson's result: all the labels from  $A \cup D \cup C$  are smaller than all the labels from  $B \cup E$ , and all the labels from  $A \cup B \cup C$  (resp.  $A \cup B$ ) occur in  $q$  before all the labels from  $D \cup E$  (resp.  $C \cup D \cup E$ ). Figure 10 shows this stage of construction for  $q = (827354619)$  whose graph is presented in Figure 9.

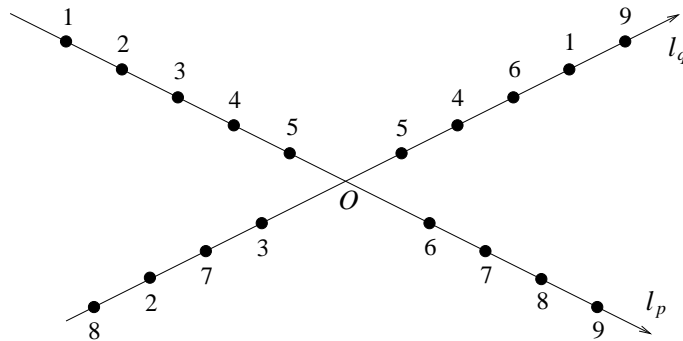


Fig. 10. Constructing of the realization of  $\{(123456789), (827354619)\}$ .

For each  $j \in [n]$ , define  $A_j = \text{conv}(P_j, Q_j)$ . Then the segments  $A_1, \dots, A_n$  are disjoint and they give a realization of  $\{p, q\}$  with transversal lines  $l_p, l_q$ .

### Proof of $2 \Leftrightarrow 3 \Leftrightarrow 4$ in Theorem 10

Stankova [13] proved that a permutation  $q = (b_1, b_2, \dots, b_n)$  is decomposable if and only if for each  $A \subseteq [n]$ ,  $|A| = 4$ , the restriction  $a|_A$  is decomposable. Moreover, the only non-decomposable permutations of  $\{1, 2, 3, 4\}$  are  $\pm(2143)$  (it is easy to verify this directly). Thus a permutation  $q = (b_1, b_2, \dots, b_n)$  is decomposable if and only if it does not contain a subpermu-

tation isomorphic to  $\pm(2143)$ , i.e., of the form  $\pm(yxtz)$  where  $x < y < z < t$ , in other words – if and only if each 4-restriction  $\{p, q\}$  is realizable. ■

## Remarks

- (1) Finding a linear (in terms of  $n$ ) algorithm which determines whether a given permutation of  $n$  is decomposable, is an easy exercise. Therefore it may be determined in linear time whether a given pair of permutations is realizable.
- (2) It can be asked whether there is a Helly-type result similar to  $1 \Leftrightarrow 4$  in Theorem 10 for more than two permutations. Namely, is it true that for each  $k$  there is a number  $h = h(k)$  so that: if a set  $\mathcal{P}$  of  $k$  u-permutations of  $[n]$  ( $n \geq h$ ) is non-realizable then it has a non-realizable  $h$ -restriction. By Theorem 5, if  $\mathcal{P}$  is non-suballowable (and therefore non-realizable), then it has a non-suballowable (and therefore non-realizable) 6-restriction. However, we think that for suballowable sets, no such  $h$  exists in general.

Given a set of permutations, it is clear that pairwise realizability is another necessary condition for their realizability. However, it is also not sufficient. In view of the mentioned above result by Edelsbrunner and Sharir (a planar family of  $n$  disjoint convex sets admits at most  $2n - 2$  geometric permutations), one way to see it is to present a pairwise realizable set with more than  $2n - 2$  members. This is done in the following example due to Abbott and Katchalski [7].

**Example** Let  $n \in \mathbb{N}$ . For each  $A \subseteq \{2, 3, \dots, \lfloor n/2 \rfloor\}$ , define the permutation  $p = p(A) = (a_1, a_2, \dots, a_n)$  on  $[n]$  as follows:

If  $j \in A$  or  $n - j + 1 \in A$ , then  $a_j = n - j + 1$ ; otherwise  $a_j = j$ .

For example, for  $n = 6$ , we have:

$$p(\emptyset) = (123456), p(\{2\}) = (153426), p(\{3\}) = (124356), p(\{2, 3\}) = (154326).$$

In this way we obtain a set  $\mathcal{P}$  of  $2^{\lfloor n/2 \rfloor - 1}$  permutations. Since all of them begin with 1, no two of them are reverses of each other. Every two of them form a realizable pair: for  $p(A)$  and  $p(B)$ , where  $A, B \subseteq \{2, 3, \dots, \lfloor n/2 \rfloor\}$ , let  $K = \{i \mid i \in A \Delta B \text{ or } n - i + 1 \in A \Delta B\}$ ,  $M = [n] \setminus K$  (here  $\Delta$  denotes symmetric difference). Then  $p(A)|_K = p(B)|_K$  and  $p(A)|_M = -p(B)|_M$ , and thus the pair  $\{p(A), p(B)\}$  is realizable by Theorem 11.

Thus pairwise realizability does not imply realizability.

### 5.3 Further observations and examples

In two previous subsections, we found two necessary conditions on a set  $\mathcal{P}$  of u-permutations for realizability by planar geometric permutations:  $\mathcal{P}$  must be suballowable and pairwise realizable. We also showed that each of these conditions is not sufficient for realizability. In this section we shall show that these conditions are in fact independent, and that they *together* still do not imply realizability.

#### 5.3.1 The necessary conditions are independent

It is easy to check that the set of u-permutations  $\{\langle 1234 \rangle, \langle 1342 \rangle, \langle 1423 \rangle\}$  is pairwise realizable but not suballowable.

Several examples of suballowable sets which are not pairwise realizable will follow from the following proposition.

**Proposition 12** *Every simple allowable sequence of permutations of  $[n]$ , with  $n \geq 5$ , contains a pair of terms of the [“forbidden”] form  $p_i = (\dots x \dots y \dots z \dots t \dots)$ ,  $p_j = \pm(\dots y \dots x \dots t \dots z \dots)$ .*

**Proof** Consider a simple allowable sequence  $\mathcal{L}$  of permutations of  $[n]$ , defined by a half-period  $p_1 \rightarrow p_2 \rightarrow \dots \rightarrow p_k [\rightarrow -p_1]$  (where  $k = \binom{n}{2}$ ), and suppose that it does not contain a forbidden pair of terms. For  $i \in [k]$ , denote the move  $p_i \rightarrow p_{i+1}$  by  $\pi_i$ . Let  $x(\pi_i) = a$  denote the fact that this move was done by the switch of labels that were in positions  $a, a+1$  in  $p_i$ . It is easy to see that  $x(\pi_{i+1}) = x(\pi_i) \pm 1$ :  $x(\pi_{i+1}) = x(\pi_i)$  is clearly impossible, and if  $|x(\pi_{i+1}) - x(\pi_i)| > 1$ , then the pair  $p_i, p_{i+2}$  is forbidden, see Figure 11.

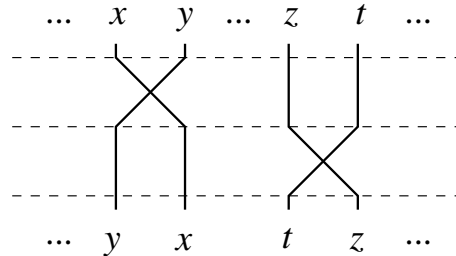


Fig. 11.  $|x(\pi_{i+1}) - x(\pi_i)| > 1$  yields a forbidden pair.

Therefore a half-period of a simple allowable sequence without a forbidden pair may be defined by  $x(\pi_1)$  and a sequence of signs  $+$  and  $-$ : for  $i \geq 2$  we write  $+$  if  $x(\pi_{i+1}) = x(\pi_i) + 1$ , and we write  $-$  if  $x(\pi_{i+1}) = x(\pi_i) - 1$ . For example, the simple allowable sequence  $(1234) \rightarrow (1324) \rightarrow (1342) \rightarrow (1432) \rightarrow (4132) \rightarrow (4312) [\rightarrow (4321)]$  is defined by  $2 + - - ++$ .

We observe that in this sequence,  $\cdots + \underbrace{- - \cdots -}_{m \text{ times}} + \cdots$  or  $\cdots - \underbrace{+ + \cdots +}_{m \text{ times}} - \cdots$  with  $m \neq 2$  is impossible. This is true since if  $m = 1$  then we have a pair of labels that switches twice (see Figure 12 (i)), and if  $m \geq 3$  then we have a forbidden pair of terms (see Figure 12 (ii)).

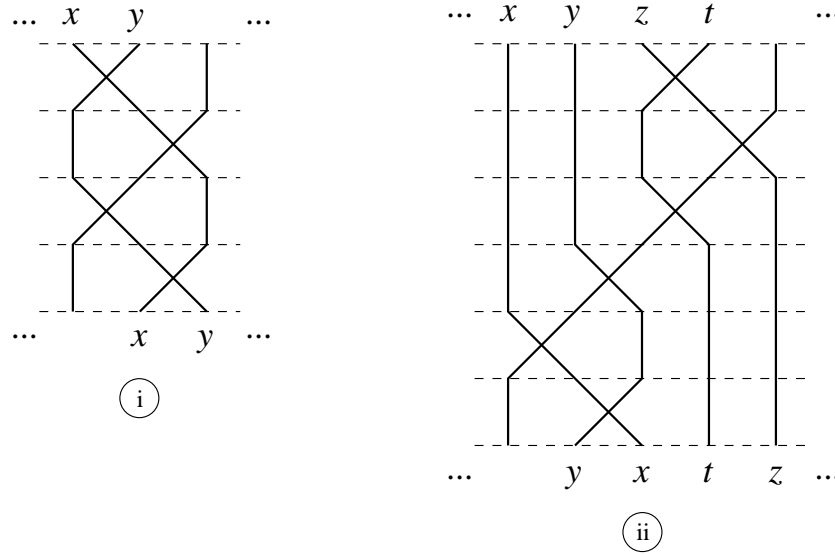


Fig. 12. Situations impossible in simple allowable sequences without  $(xyzt), \pm(yxtz)$ .

The situation  $\cdots + - - + \cdots$  (resp.  $\cdots - + + - \cdots$ ) does not give an immediate contradiction. However it follows by an analogous argument that it can not be extended to  $\cdots + - - + + \cdots + - \cdots$  (resp.  $\cdots - + + - - \cdots - + \cdots$ ), i.e., the sign cannot change once again.

Thus in the sequence of signs (we can also assume without loss of generality that the first sign after  $\pi_1$  is  $+$ ), the sign either does not change ( $++\cdots+$ ), or changes once ( $++\cdots+-\cdots-$ ), or changes twice (with  $-$  only in two consecutive positions:  $++\cdots+- - + + \cdots +$ ).

It follows the  $\mathcal{L}$  has at most  $2n-2$  terms, which is less than  $\binom{n}{2}$ , a contradiction.  $\blacksquare$

**Remark and Conjecture** Proposition 12 shows that the condition of pairwise realizability (avoiding a pair of terms of the form  $(\dots x \dots y \dots z \dots t \dots)$ ,  $\pm(\dots y \dots x \dots t \dots z \dots)$ ) in a suballowable sequence is quite restrictive in the sense that it “prevents” the sequence from being very long. Indeed, we could not find an example of such a sequence with more than  $2n - 2$  terms (recall that this is the bound on number of geometric permutations of a planar family of  $n$  sets). The following example that attains this length was found by Katchalski, Lewis and Zaks [9]:

$$p_1 = (123 \dots n).$$

For  $i = 1, \dots, n - 1$ : in the move  $p_i \rightarrow p_{i+1}$  the label 1 switches with (its right neighbor)  $i + 1$ .

For  $i = n, \dots, 2n - 3$ : in the move  $p_i \rightarrow p_{i+1}$  the label  $n$  switches with (its left neighbor)  $2n - i - 1$ .

This example of a suballowable sequence of  $2n - 2$  pairwise realizable permutations of  $[n]$  is not unique: here is a non-isomorphic example:

$p_1 = (123 \dots n)$ .

For  $i = 1, \dots, n - 1$ : in the move  $p_i \rightarrow p_{i+1}$  the label 1 switches with (its right neighbor)  $i + 1$ .

In the move  $p_n \rightarrow p_{n+1}$  the label  $n - 1$  switches with  $n$ .

In the move  $p_{n+1} \rightarrow p_{n+2}$  the label  $n - 2$  switches with  $n$ .

In the move  $p_{n+2} \rightarrow p_{n+3}$  the label  $n - 2$  switches with  $n - 1$ .

For  $i = 3, \dots, n - 3$ : in the move  $p_{n+i} \rightarrow p_{n+i+1}$  the label  $n - i$  switches with the string  $[n(n - 1)(n - 2) \dots (n - i + 1)]$ .

Figure 13 presents wiring diagrams of these sequences, for  $n = 8$ .

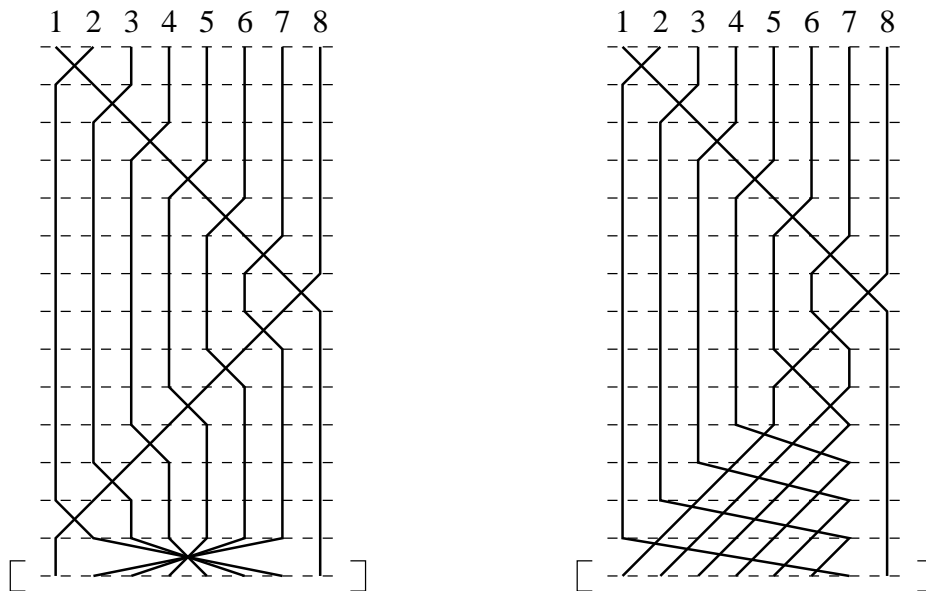


Fig. 13. Two suballowable sequences of pairwise realizable permutations of  $\{1, \dots, 8\}$ , of length 14.

We conjecture that a suballowable sequence of pairwise realizable permutations has at most  $2n - 2$  terms. If true, this would give a new proof of Edelsbrunner-Sharir theorem.

5.3.2 The necessary conditions together are not sufficient

The next example shows that two necessary conditions (suballowability and pairwise realizability), *together*, are not sufficient for realizability. Note that the sequence of permutations is the same one that we had in Proposition 8.

**Proposition 13** *The suballowable sequence  $\mathcal{L}$  with half-period  $(123456) \rightarrow (412563) \rightarrow (541632)[\rightarrow (654321)]$  is not realizable by geometric permutations in  $\mathbb{R}^2$ .*

We shall use the following simple fact.

**Observation 14** *Let  $p = (123)$ ,  $q = (132)$  be two directed geometric permutations of a family  $\{A_1, A_2, A_3\}$  of disjoint convex sets, induced by directed transversal lines  $l_1$  and  $l_2$ . Then  $A_1$  meets both  $l_1$  and  $l_2$  before their point of intersection  $O$ .*

**Proof of Observation 14** Denote by  $l_1^-$  (resp.  $l_1^+$ ) the open ray consisting of the points of  $l_1$  before (resp. after)  $O$ ; similarly for  $l_2^-$  and  $l_2^+$ .

We need to show that  $A_1 \cap l_1 \subset l_1^-$  and that  $A_1 \cap l_2 \subset l_2^-$ .

Suppose  $A_1 \cap (l_1^+ \cup \{O\}) \neq \emptyset$ . Then  $A_2 \cap l_1 \subset l_1^+$  and  $A_3 \cap l_1 \subset l_1^+$ .

It is clear that  $O \notin A_3$ . If  $A_3 \cap l_2 \subset l_2^+$  then also  $A_2 \cap l_2 \subset l_2^+$ , and  $A_3 \cap A_2 \neq \emptyset$ . If  $A_3 \cap l_2 \subset l_2^-$  then also  $A_1 \cap l_2 \subset l_2^-$ , and  $A_3 \cap A_1 \neq \emptyset$  (see Figure 14).

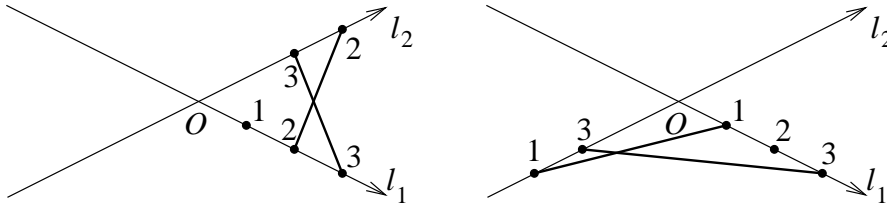


Fig. 14. Illustration to the proof of Observation 14.

Thus  $A_1 \cap l_1 \subset l_1^-$ . By symmetry, also  $A_1 \cap l_2 \subset l_2^-$ . ■

**Proof of Proposition 13** Suppose  $\mathcal{F} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$  realizes  $\mathcal{L}$  with directed transversal lines  $l_1, l_2, l_3$  which induce the permutations according to the order in which they appear in  $\mathcal{L}$ . Since  $\mathcal{L}$  is a suballowable sequence, we conclude that, without loss of generality,  $(l_1, l_2, l_3)$  is the order in which these lines appear on the circle of directions, see Figure 15.

For  $i, j \in \{1, 2, 3\}$  denote  $O_{ij} = O_{ji} = l_i \cap l_j$ . If the points  $O_{12}, O_{13}, O_{23}$  are distinct, there are two possible cases: the first is  $l_1 : (O_{12} \prec O_{13}), l_2 : (O_{21} \prec O_{23}), l_3 : (O_{31} \prec O_{32})$ ; the second is  $l_1 : (O_{13} \prec O_{12}), l_2 : (O_{23} \prec O_{21}), l_3 : (O_{32} \prec O_{31})$ , see Figure 16.

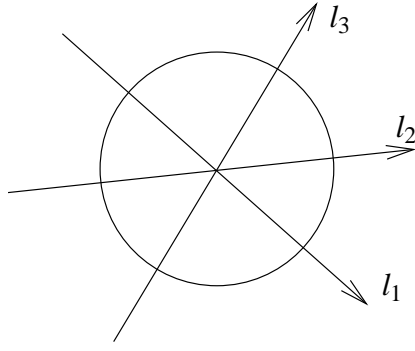


Fig. 15. The order of the lines  $l_1, l_2, l_3$  on the circle of directions.

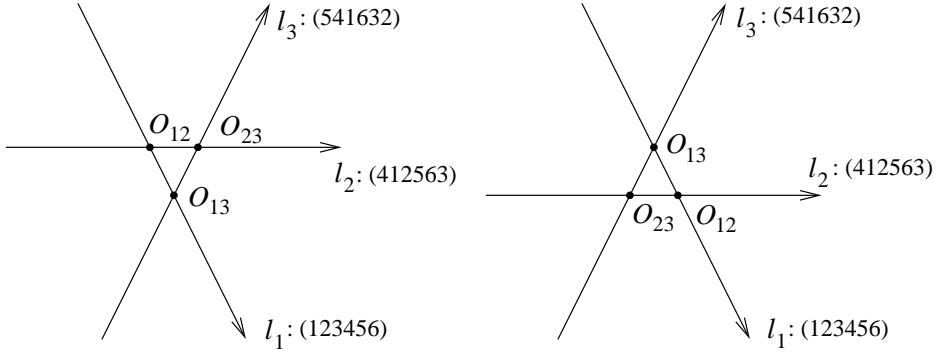


Fig. 16. Two cases for the lines  $l_1, l_2, l_3$ .

However, the second case can be obtained from the first by relabeling the sets  $1 \mapsto 4, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 5, 5 \mapsto 6, 6 \mapsto 3$ , the transversals  $l_1 \mapsto l_2, l_2 \mapsto l_3, l_3 \mapsto -l_1$ , and the intersection points  $O_{12} \mapsto O_{23}, O_{13} \mapsto O_{12}, O_{23} \mapsto O_{13}$ . Therefore we shall consider only the first case (together with the case when the points  $O_{12}, O_{13}, O_{23}$  coincide).

We claim that

$$l_1 : 1 \prec 2 \prec 3 \prec O_{12} \preceq O_{13} \prec 4 \prec 5 \prec 6,$$

$$l_2 : 4 \prec 1 \prec 2 \prec O_{21} \preceq O_{23} \prec 5 \prec 6 \prec 3,$$

$$l_3 : 5 \prec 4 \prec 1 \prec O_{31} \preceq O_{32} \prec 6 \prec 3 \prec 2 \text{ (see Figure 17).}$$

In order to prove this, we use Observation 14 several times:

Since  $l_1 : 3 \prec 4 \prec 5$  and  $-l_2 : 3 \prec 5 \prec 4$ , we have by Observation 14  $l_1 : 3 \prec O_{12}$ . Therefore  $l_1 : 1 \prec 2 \prec 3 \prec O_{12}$ . Similarly we obtain  $l_2 : O_{23} \prec 5 \prec 6 \prec 3$  and  $l_3 : 5 \prec 4 \prec 1 \prec O_{31}$ .

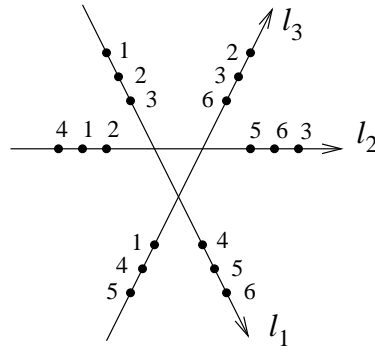


Fig. 17. Position of the sets  $A_i$  along the lines  $l_j$ , relative to the points of intersection of the transversals.

Since  $-l_1 : 4 \prec 2 \prec 1$  and  $l_3 : 4 \prec 1 \prec 2$ , we have by Observation 14  $l_1 : O_{13} \prec 4$ . Therefore  $l_1 : O_{13} \prec 4 \prec 5 \prec 6$ . Similarly we obtain  $l_2 : 4 \prec 1 \prec 2 \prec O_{21}$  and  $l_3 : O_{32} \prec 6 \prec 3 \prec 2$ .

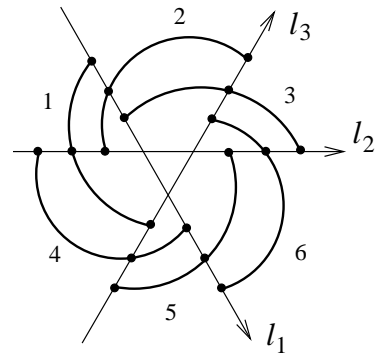


Fig. 18. A realization of  $(123456) \rightarrow (412563) \rightarrow (541632)$  by arcs.

Using an affine transformation, we can assume that the angle between every two transversals is  $60^\circ$ . Then the realization must look similar to the situation in Figure 18, with arcs replaced by convex sets. We can also assume that the sets are segments with endpoints on transversal lines: let  $P \in A_1 \cap l_1$ ,  $Q \in A_1 \cap l_3$ . Replace  $A_1$  by the segment  $PQ$ . This argument can be applied to all the sets in  $\mathcal{F}$ , without changing the permutations induced by  $l_1, l_2, l_3$ .

However, there is no realization of  $\mathcal{L}$  with segments with endpoints on transversal lines. Suppose that  $\mathcal{F}$  is such a realization, and denote by  $\delta$  the minimal distance between an endpoint of one of the segments and one of the points of intersection of the transversals. Without loss of generality,  $\delta = \text{dist}(A_1 \cap l_3, O_{13})$  or  $\delta = \text{dist}(A_2 \cap l_2, O_{21})$  (other possibilities can be obtained from one of these by relabeling the sets and the transversals). Then (since all the angles between the transversals are  $60^\circ$ ) in the first case  $\text{dist}(A_2 \cap l_2, O_{21}) \geq \delta$  and  $A_1$  misses  $l_1$ ; in the second case  $\text{dist}(A_3 \cap l_1, O_{12}) \geq \delta$  and  $A_2$  misses  $l_3$  (see Figure 19). It follows that  $\mathcal{L}$  is non-realizable by geometric permutations in the plane. ■

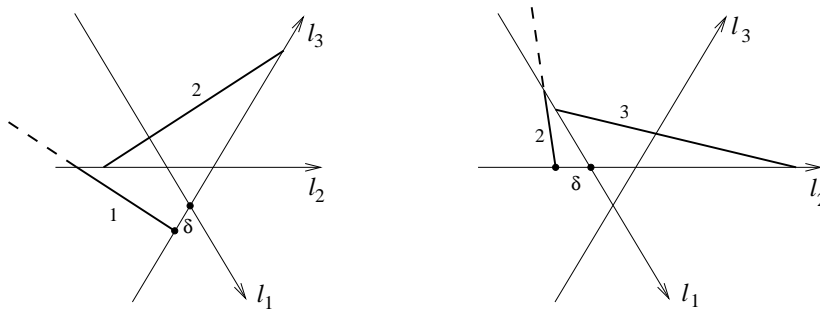


Fig. 19.  $(123456) \rightarrow (412563) \rightarrow (541632)$  is non-realizable by planar geometric permutations.

**Remark** A similar construction was used by Sharir and Smorodinsky in their study of neighbors in geometric permutations [12]. Though used in different contexts, both constructions point out constraints in sets of permutations realizable by planar geometric permutations.

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