Geometric Permutations in the Plane
and in Euclidean Spaces of Higher Dimension

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Geometric Permutations in the Plane and in Euclidean Spaces of Higher Dimension

Research Thesis

Submitted in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy

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Submitted to the Senate of the Technion — Israel Institute of Technology
Shevat 5765 Haifa February 2005

This research was carried out under the supervision of Prof. Meir Katchalski at the Faculty of Mathematics.

I would like to express my gratitude and appreciation to my advisor, Prof. Meir Katchalski, for his support throughout the course of my research.

I would also like to thank
Prof. Roy Meshulam for the instructive discussions I've had with him,
Prof. Ron Aharoni, for his useful advice,
and Prof. Micha Sharir, for his comments on the first draft of my thesis.

Finally, I would like to thank all the people at the Technion's Department of Mathematics for the pleasant atmosphere that has accompanied my studies and research.

The generous financial help of the Technion is gratefully acknowledged.

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Abstract

This work deals with geometric permutations for families of disjoint convex sets in Euclidean spaces.

Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^d$. A straight line l is a *transversal* of $\mathcal{F}$ if it intersects every member of $\mathcal{F}$. Each transversal intersects the members of $\mathcal{F}$ in an order which can be described by a pair of permutations of $[n] = \{1, 2, \ldots, n\}$, which are reverses of each other. Such a pair is called a *geometric permutation* (the two permutations that form it are referred to as *representatives* of the geometric permutation).

The study of geometric permutations was used in solving several problems in Geometric Transversal Theory.

In this work we obtain some new results on geometric permutations.

The following result refutes a conjecture by Katchalski, Lewis and Liu, showing that the maximal number of geometric permutations in a family of disjoint translates of a convex set is not bounded by a constant:

- For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in $\mathbb{R}^3$ and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of X that admits at least $n + 1$ geometric permutations.

A family $\mathcal{P}$ of permutations of n sets is *realizable* in $\mathbb{R}^d$ if there exists a family $\mathcal{F}$ of n disjoint convex sets in $\mathbb{R}^d$ that has all the members of $\mathcal{P}$ as geometric permutations; otherwise $\mathcal{P}$ is *non-realizable*, or *forbidden*, in $\mathbb{R}^d$.

We find for which values of k there are families of k permutations forbidden in $\mathbb{R}^d$:

- If $d \geq 2k - 1$ then each family of k permutations is realizable in $\mathbb{R}^d$.
- If $d \leq 2k - 2$ then there is a family of k permutations which is forbidden in $\mathbb{R}^d$.

We study families of permutations realizable and non-realizable in $\mathbb{R}^2$. We find necessary and sufficient conditions for realizability in $\mathbb{R}^2$ for families of two permutations $\{p, q\}$

- A family $\{p, q\}$ of permutations of $[n]$ is forbidden in $\mathbb{R}^2$ if and only if there are $i, j, k, l \in [n]$ so that $p = \langle \ldots i \ldots j \ldots k \ldots l \ldots \rangle$, $q = \langle \ldots j \ldots i \ldots l \ldots k \ldots \rangle$.

- A family $\{p = (1, 2, \dots, n), q = (q_1, q_2, \dots, q_n)\}$ of permutations for $[n]$ is realizable in $\mathbb{R}^2$ if and only if q is decomposable to an ascending and a descending sub-permutations.

- A family $\{p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n)\}$ of two permutations for $[n]$ is realizable in $\mathbb{R}^2$ if and only if $[n]$ can be partitioned into two subsets so that the members of one part appear in the same order in p and q, and the members of the second part appear in opposite orders in p and q.

We provide a linear algorithm which checks whether a given family of two permutations of $[n]$ is realizable in $\mathbb{R}^2$.

Then we study realizability in $\mathbb{R}^2$ of families of permutations with more than two members. We find two necessary conditions for realizability:

- Let $\mathcal{P} = \{\tilde{p}^1, \tilde{p}^2, \dots, \tilde{p}^k\}$ be a family of permutations realizable in $\mathbb{R}^2$. Then:

  1. No two of them form a forbidden pair.
  2. It is possible to choose a representative p^j of each $\tilde{p}^j$ and to order them, $p^{i_1} \ll p^{i_2} \ll \cdots \ll p^{i_k}$, so that there are no $x, y, z \in [k]$, $x < y < z$, so that for some $a, b \in [n]$, $p^{i_x} : (a \prec b)$, $p^{i_y} : (b \prec a)$, $p^{i_z} : (a \prec b)$.

We give an example which shows that these necessary conditions together are not sufficient: the family of permutations $\mathcal{P} = \{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$ satisfies both the conditions, but it is non-realizable in $\mathbb{R}^2$.

Chapter 1

Introduction

1.1 Background on Geometric Permutations

This work deals with geometric permutations for families of disjoint convex sets in Euclidean spaces $\mathbb{R}^d$.

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^d$. A straight line l is a *transversal* of $\mathcal{F}$ if it intersects every member of $\mathcal{F}$. Each transversal intersects the members of $\mathcal{F}$ in an order which can be described by an *undirected permutation*, i.e. a pair of permutations of $[n] = \{1, 2, \ldots, n\}$, which are reverses of each other. Such a pair is called a *geometric permutation*.

We shall use the following notations. If a directed transversal l meets the sets in a certain order, this order will be denoted by $\prec$, and we shall write $l : (A_{x_1} \prec A_{x_2} \prec \cdots \prec A_{x_n})$, or just $l : (x_1 \prec x_2 \prec \cdots \prec x_n)$, where $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$. This order is described by the permutation $p = (x_1, x_2, \ldots, x_n)$. Then $-l$ denotes the same line oriented conversely: $-l : (A_{x_n} \prec A_{x_{n-1}} \prec \cdots \prec A_{x_1})$, and $-p$ denotes the reverse of the permutation p: $-p = (x_n, x_{n-1}, \ldots, x_1)$. If l is not directed, we shall denote the order by $*$ and we shall write $l : (A_{x_1} * A_{x_2} * \cdots * A_{x_n})$, or just $l : (x_1 * x_2 * \cdots * x_n)$, and denote the corresponding geometric permutation by $\tilde{p} = \{p, -p\} = \{(x_1, x_2, \ldots, x_n), (x_n, x_{n-1}, \ldots, x_1)\} = \langle x_1, x_2, \ldots, x_n \rangle$. The permutations p, $-p$ are *representatives* of $\tilde{p}$. Where this is not essential, we shall not distinguish between geometric permutations and their representatives. Figure 1.1 presents a planar family of four sets A_1, A_2, A_3, A_4 with six geometric permutations: $\langle 3412 \rangle$, $\langle 3142 \rangle$, $\langle 1342 \rangle$, $\langle 1432 \rangle$, $\langle 1423 \rangle$, $\langle 1243 \rangle$.

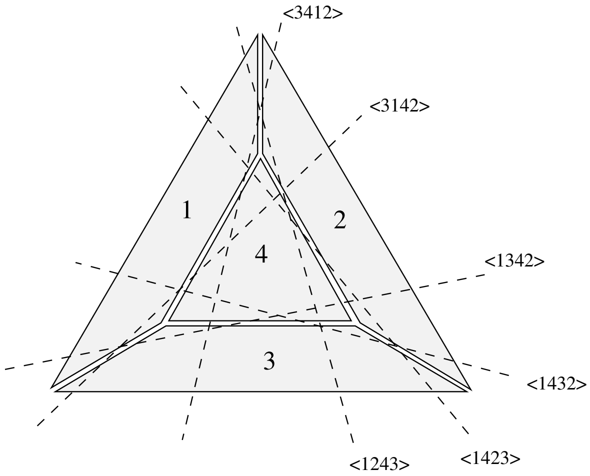

Figure 1.1: A family of four sets that admits six geometric permutations.

Geometric permutations were introduced by Katchalski, Lewis and Liu [16] as a tool for dealing with problems in Geometric Transversal Theory. The following two Helly-type results were proved using geometric permutations.

- Let $\mathcal{F}$ be a family of pairwise disjoint translates of a convex set in $\mathbb{R}^2$. If every 5 members of $\mathcal{F}$ have a common transversal, then the whole family $\mathcal{F}$ has a common transversal (Tverberg, 1989 [26]).

This theorem confirms Grünbaum's conjecture from 1958 [11]. The Helly number 5 is the best possible. A weaker version of this result (with 128 instead 5) was proved by Katchalski in 1986 [15].

- Let $\mathcal{F}$ be a finite family of pairwise disjoint unit balls in $\mathbb{R}^3$. If every 11 members of $\mathcal{F}$ have a common transversal, then the whole family $\mathcal{F}$ has a common transversal (Cheong, Goaoc and Holmsen, 2005 [5]).

It is not known whether the Helly number 11 is the best possible. A weaker version of this result (with Helly number 46) was proved by Holmsen, Katchalski and Lewis in 2003 [13], and then it was improved (with Helly number 18) by Cheong, Goaoc and Na in 2004 [6]. There is no Helly-type result of this type with transversal lines for disjoint translates of a general set in $\mathbb{R}^3$ (Holmsen and Matoušek, 2001, [14]).

Survey articles on Geometric Transversal Theory [7, 8, 10, 12, 27, 29] contain other generalizations and related results.

1.2 Results presented in this work

Several papers on geometric permutations deal with a maximal number of geometric permutations that a family of n disjoint convex sets in $\mathbb{R}^d$ can admit. The known bounds are the following:

- The maximal number of geometric permutations for families of n disjoint convex sets in $\mathbb{R}^2$ is $2n-2$ (for $n \geq 4$) (as an upper bound: Edelsbrunner and Sharir, 1990 [9]; as a lower bound: Katchalski, Lewis and Zaks, 1985 [19]).
- The maximal number of geometric permutations for families of n disjoint convex sets in $\mathbb{R}^d$ is $O(n^{2d-2})$ and $\Omega(n^{d-1})$ (the upper bound: Wenger, 1990 [28]; the lower bound: Katchalski, Lewis and Liu, 1992 [18]).
- The maximal number of geometric permutations for families of n disjoint balls in $\mathbb{R}^d$ is $\Theta(n^{d-1})$. (Smorodinsky, Mitchell and Sharir, 2000 [25]).

Several results deal with geometric permutations for families of disjoint convex sets with several restrictions on sets or on transversal lines involved:

- The maximal number of geometric permutations for families of n fat disjoint sets in $\mathbb{R}^d$ is $\Theta(n^{d-1})$ (Katz and Varadarajan, 2001 [21]).

  (A *fat set* is a set with the bounded ratio of the radius of the smallest ball containing the set to the radius of the largest ball contained in the set.)
- The maximal number of geometric permutations for families of n disjoint convex sets in $\mathbb{R}^d$, induced by lines passing through a fixed point, is $\Theta(n^{d-1})$ (Aronov and Smorodinsky, 2004 [2]).

In particular, several results deal with families of disjoint translates of a convex set:

- The maximal number of geometric permutations for families of n disjoint translates of a convex set in $\mathbb{R}^2$ is 3 (for $n \geq 3$) (Katchalski, Lewis and Liu, 1987-92 [17, 18]).

It was conjectured [18] that similar results exist in higher dimensions as well. That is, for each natural d there exists a constant number $\alpha = \alpha(d)$ such that the maximal number of geometric permutations for families of n disjoint translates of any convex set in $\mathbb{R}^d$ is α. It was found that such a number exists for families of disjoint congruent balls:

- For each natural d, the maximal number of geometric permutations for families of n disjoint congruent balls in $\mathbb{R}^d$ is 2 (for $n \geq 9$) (Cheong, Goaoc and Na, 2004 [6]).

In Chapter 2 we disprove the mentioned above conjecture, proving that there is no such result for a general set in $\mathbb{R}^3$ (and therefore in $\mathbb{R}^d$ for each $d \geq 3$):

Theorem 2.1. *For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in $\mathbb{R}^3$ and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of X that admits at least $n + 1$ geometric permutations.*

Then we show that a stronger result holds: the same can be done with a set that does not depend on n:

Theorem 2.2. *There exists a convex set Y in $\mathbb{R}^3$ such that for each $n \in \mathbb{N}$, $n > 1$, there exists a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of Y that admits at least $n + 1$ geometric permutations.*

Chapters 3 and 4 deal with another aspect of geometric permutations. Let $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ be a family of permutations for n sets. If there is a family $\mathcal{F}$ of disjoint convex sets in $\mathbb{R}^d$ that has all the members of $\mathcal{P}$ as geometric permutations, we say that $\mathcal{P}$ is *realizable* in $\mathbb{R}^d$, otherwise we say that $\mathcal{P}$ is *non-realizable*, or *forbidden*, in $\mathbb{R}^d$. It was noted in the earliest paper on geometric permutations [16] that the family of permutations $\{\langle 1234\rangle, \langle 2143\rangle\}$ is forbidden in $\mathbb{R}^2$. We study the property of families of permutations of being realizable or forbidden in $\mathbb{R}^d$.

In Chapter 3 we find a necessary and sufficient condition on the relation between d and k for the existence of a forbidden family of k permutations (of some set $[n]$) in $\mathbb{R}^d$:

Theorem 3.1. *If $d \geq 2k - 1$ then each family of k permutations is realizable in $\mathbb{R}^d$.*

Theorem 3.2. *If* $d \leq 2k - 2$ *then there is a family of* k *permutations which is forbidden in* $\mathbb{R}^d$.

In Chapter 4 we investigate the realizability of families of permutations in $\mathbb{R}^2$. In Sections 4.2 – 4.3 we find a necessary and sufficient condition for realizability of families of two permutations $\{p, q\}$. The condition is that no restriction of p, q to some 4 sets results in a family isomorphic to the mentioned above "forbidden pair" $\{\langle 1234\rangle, \langle 2143\rangle\}$:

Theorem 4.9 *A family* $\{p, q\}$ *of permutations for* $[n]$ *is forbidden in* $\mathbb{R}^2$ *if and only if there are* $i, j, k, l \in [n]$ *so that* $p = \langle \ldots i \ldots j \ldots k \ldots l \ldots \rangle$, $q = \langle \ldots j \ldots i \ldots l \ldots k \ldots \rangle$.

In order to prove this, we show a correspondence between realizable families of two geometric permutations and permutations that can be partitioned into an ascending and a descending sub-permutations:

Theorem 4.7 *A family* $\{p = (1, 2, \ldots, n), q = (q_1, q_2, \ldots, q_n)\}$ *of permutations for* $[n]$ *is realizable in* $\mathbb{R}^2$ *if and only if* q *is decomposable to an ascending and a descending sub-permutations.*

This result can be generalized (by relabeling the sets) to a more general statement:

Corollary 4.8 *A family* $\{p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n)\}$ *of two permutations for* $[n]$ *is realizable in* $\mathbb{R}^2$ *if and only if* $[n]$ *can be partitioned into two subsets so that the members of one part appear in the same order in* p *and* q*, and the members of the second part appear in opposite orders in* p *and* q.

In Section 4.4 we provide a linear-time (in terms of n) algorithm which checks whether a given family of two permutations is realizable in $\mathbb{R}^2$.

In Section 4.5 we study realizability in $\mathbb{R}^2$ of families of permutations with more than two members. We give two necessary conditions on $\mathcal{P} = \{\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^k\}$ for being realizable in $\mathbb{R}^2$ (first of them is trivial):

Theorems 4.11 and 4.12. Let $\mathcal{P} = \{\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^k\}$ be a family of permutations realizable in $\mathbb{R}^2$. Then:

1. No two of them form a forbidden pair.
2. It is possible to choose a representative p^j of each $\tilde{p}^j$ and to order them,

$p^{i_1} \ll p^{i_2} \ll \cdots \ll p^{i_k}$, so that there are no $x, y, z \in [k]$, $x < y < z$, so that for some $a, b \in [n]$, $p^{i_x} : (a \prec b)$, $p^{i_y} : (b \prec a)$, $p^{i_z} : (a \prec b)$.

We show that these necessary conditions are independent. However, even together they are not sufficient: in Section 4.6 we give an example of a family of three permutations for six sets which satisfies both the conditions but is forbidden in $\mathbb{R}^2$:

Claim 4.13. $\mathcal{P} = \{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$ *is not realizable in* $\mathbb{R}^2$.

All the chapters are concluded by a discussion on related open problems and suggestions for a future research.

Chapter 2

The maximal number of geometric permutations for n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$

In this chapter[1] we deal with geometric permutations for families of disjoint translates of a convex set. Several results are known for such families. Katchalski, Lewis and Liu proved that for such families in $\mathbb{R}^2$, the maximal number of geometric permutations is 3 [17, 18]. They also conjectured [18] that for each natural d, there is a *constant* upper bound on the number of geometric permutations for such families in $\mathbb{R}^d$ (the conjectured bound was $\frac{(d+1)!}{2}$). However, the only known upper bound in $\mathbb{R}^d$ is $O(n^{d-1})$ (this follows from [21]). A constant upper bound is known in a special case: for families of congruent balls in $\mathbb{R}^d$ [20] (improved in [6]; the bound is 2 when $n \geq 9$).

We refute the conjecture mentioned above by proving that there is no constant bound for the maximal number of geometric permutations of translates of a general set:

Theorem 2.1 *For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in $\mathbb{R}^3$ and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of X that admits at least $n + 1$ geometric permutations.*

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[1] The results from this chapter appear in the paper *The maximal number of geometric permutations for n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$* [4], accepted for publication in *Discrete and Computational Geometry.*

In Section 2.2 we show that a stronger results holds: there is a set Y that does not depend on n and satisfies Theorem 2.1:

Theorem 2.2 *There exists a convex set Y in $\mathbb{R}^3$ such that for each $n \in \mathbb{N}$, $n > 1$, there is a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of Y that admits at least $n+1$ geometric permutations.*

Our constructions use an idea of Holmsen and Matoušek. They showed [14] that in $\mathbb{R}^3$ there is no Helly-type theorem analogous to Tverberg's result mentioned in the Introduction (the solution of Grünbaum's conjecture, see page 4). For each $n \in \mathbb{N}$ they construct a family of disjoint translates of a convex set such that each n members of the family have a transversal line, while the entire family does not. In general, the idea of their construction is to take first a family of disjoint sets that have the desired transversal properties, but are not translates of each other, and then to append them one to another in order to obtain translates, preserving their disjointness and transversal properties. Both constructions use the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. This surface has been used earlier for the construction of several examples with translersal lines (see, for example, Theorem 2.9 by Aronov, Goodman, Pollack and Wenger, in [10], p. 171, and the construction desrcibed there).

2.1 The construction (Proof of Theorem 2.1)

Let $n \in \mathbb{N}$.

Planes and Lines

Denote by Σ the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. For each $i \in \{0, 1, \ldots, n\}$, let λ_i be the plane $y = i$, and let l_i be the line $\lambda_i \cap \Sigma = \{(x, y, z) : y = i, z = xi\}$. These $n+1$ lines will be transversal lines for $\mathcal{F}$, inducing different geometric permutations.

For each $m \in \{1, 2, \ldots, n\}$, let u_m denote the plane $x = 2mn^2$, u'_m the plane $x = 2mn^2 + 1$, and w_m the plane $x = 2mn^2 + n^2 + 1/2$. These planes will be used in the construction of $\mathcal{F}$, and in the proof of the disjointness of its members.

The set X

For each $m \in \{1, 2, \dots, n\}$, define four points on Σ as follows:

$$P_{m,1} = l_{m-1} \cap u_m = (2mn^2, m-1, 2mn^2 \cdot (m-1)),$$

$$P_{m,2} = l_m \cap u'_m = (2mn^2+1, m, (2mn^2+1) \cdot m);$$

$$Q_{m,1} = l_m \cap u_m = (2mn^2, m, 2mn^2 \cdot m),$$

$$Q_{m,2} = l_{m-1} \cap u'_m = (2mn^2+1, m-1, (2mn^2+1) \cdot (m-1)).$$

Note that $P_{m,1}, P_{m,2} \in \pi_m$, and $Q_{m,1}, Q_{m,2} \in \sigma_m$, where π_m is the plane $y = x - 2mn^2 + m - 1$ and σ_m is the plane $y = -x + 2mn^2 + m$. The planes π_m and σ_m are parallel to the planes $y = x$ and $y = -x$ respectively.

Let a_m be the segment that contains $P_{m,1}$ and $P_{m,2}$ with endpoints in the planes λ_0 and λ_n, and let b_m be the segment that contains $Q_{m,1}$ and $Q_{m,2}$ with endpoints in the planes λ_0 and λ_n. Figures 2.1 and 2.2 show a_i-s and b_i-s for $n = 3$ (In these figures, the solid parts of the segments are above Σ, and the dashed are below it. Note that the figures are not drawn to scale: in fact, the segments are much further apart).

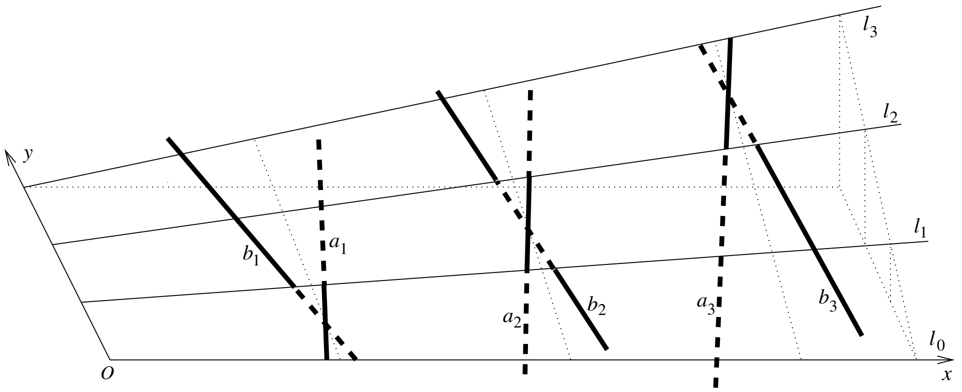

Figure 2.1: The segments a_i and b_i, for $n = 3$. The solid parts are above Σ, and the dashed are below it.

In what follows, "the highest (lowest) point" (of a set) means "the point with the maximal (minimal) z-coordinate". (This is used only when such points are unique.)

Now define two sets X^L and X^U. Each of them is a polygonal line: $X^L = \tilde{a}_1 \cup \tilde{a}_2 \cup \dots \cup \tilde{a}_n$, $X^U = \tilde{b}_1 \cup \tilde{b}_2 \cup \dots \cup \tilde{b}_n$, where each $\tilde{a}_m$ is a translate of a_m, and each $\tilde{b}_i$ is a translate of b_i, so that:

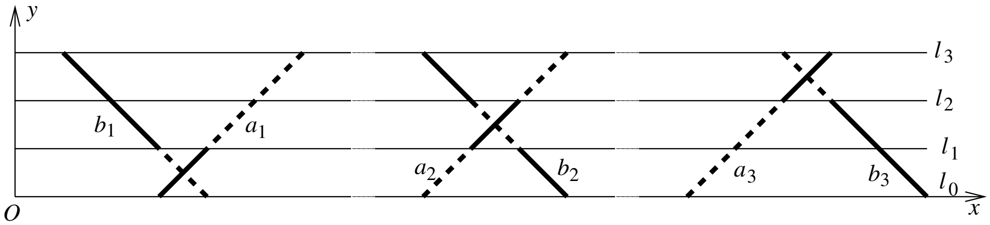

Figure 2.2: The segments a_i and b_i, for $n = 3$, viewed from above.

- the lowest point of $\tilde{a}_1$ is $(0, 0, 0)$, and for each $m \in \{2, 3, \ldots, n\}$ the lowest point of $\tilde{a}_m$ coincides with the highest point of $\tilde{a}_{m-1}$;
- the highest point of $\tilde{b}_1$ is $(0, n^2, H_L + H + H_U)$ (where H_L and H_U are the z-heights of X^L and X^U respectively, and H is a large positive number), and for each $m \in \{2, 3, \ldots, n\}$ the highest point of $\tilde{b}_m$ coincides with the lowest point of $\tilde{b}_{m-1}$.

We make some observations on X^L and X^U.

1. Each a_m lies in π_m, and each b_m lies in σ_m. It follows that X^L lies in the plane $y = x$, and X^U lies in the plane $y = -x + n^2$.
2. Each a_m contributes n to the lengths of the x- and y-projections of X^L, and each b_m contributes n to the lengths of the x- and y-projections of X^U. Thus the x- and y-projections of X^L and of X^U are $[0, n^2]$.
3. The slopes of a_m and b_m, relative to the plane $z = 0$, are, respectively, $\frac{1}{\sqrt{2}}((2mn^2 + 1)m - 2mn^2(m - 1)) = \frac{1}{\sqrt{2}}(2n^2 + 1)m$ and $\frac{1}{\sqrt{2}}((2mn^2 + 1)(m - 1) - 2m^2n^2) = -\frac{1}{\sqrt{2}}((2n^2 - 1)m + 1)$.
   This means that if $m < m'$ then the slope of a_m is smaller than that of $a_{m'}$, and the slope of b_m is smaller than that of $b_{m'}$. It follows that X^L is downward convex, and X^U is upward convex polygonal line.

Let $X = \mathrm{conv}(X^L \cup X^U)$ (see Figure 2.3; note that in fact X^U is situated high above X^L). It is clear that X is a convex set.

The family $\mathcal{F}$ of disjoint translates of X

For each $m \in \{1, 2, \ldots, n\}$, define A_m to be a translate of X such that $\tilde{a}_m$ is translated to a_m, and B_m to be a translate of X such that $\tilde{b}_m$ is translated

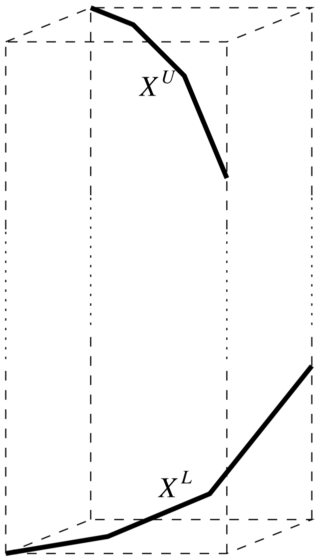

Figure 2.3: The set X for $n = 3$: the bold polygonal lines are X^L and X^U; X is their convex hull.

to b_m. Denote by A^U_m (A^L_m) the polygonal line on A_m that corresponds to X^U (X^L) on X, and by B^U_m (B^L_m) the polygonal line on B_m that corresponds to X^U (X^L) on X.

The family $\mathcal{F} = \{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}$ is a family of $2n$ translates of X. We prove that they are pairwise disjoint, and that $\mathcal{F}$ has at least $n + 1$ geometric permutations.

Disjointness of the members of $\mathcal{F}$

First, note that for each $m \in \{1, 2, \ldots, n - 1\}$, the sets A_m and B_m have points ($P_{m,1}$ and $Q_{m,1}$ respectively) with x-coordinate $2mn^2$, and the sets A_{m+1} and B_{m+1} have points ($P_{m+1,2}$ and $Q_{m+1,2}$ respectively) with x-coordinate $2(m + 1)n^2 + 1$. The x-lengths of the sets are n^2. It follows that A_m and B_m are to the left of the plane w_m (recall that this plane is $x = 2mn^2 + n^2 + 1/2$), while A_{m+1} and B_{m+1} are to its right. Hence for $m \neq m'$, $A_m \cap A_{m'}$, $A_m \cap B_{m'}$ and $B_m \cap B_{m'}$ are $\emptyset$.

It remains to prove that $A_m \cap B_m = \emptyset$ for each $m \in \{1, 2, \ldots, n\}$. Let τ_m be the plane that contains the point $(2mn^2 + 1/2, m - 1/2, (2mn^2 + 1/2)(m - 1/2))$, and parallel to a_m and b_m. We claim that this plane separates A_m from B_m. Let $t_m = \pi_m \cap \tau_m$, $s_m = \sigma_m \cap \tau_m$ (the planes π_m and σ_m were

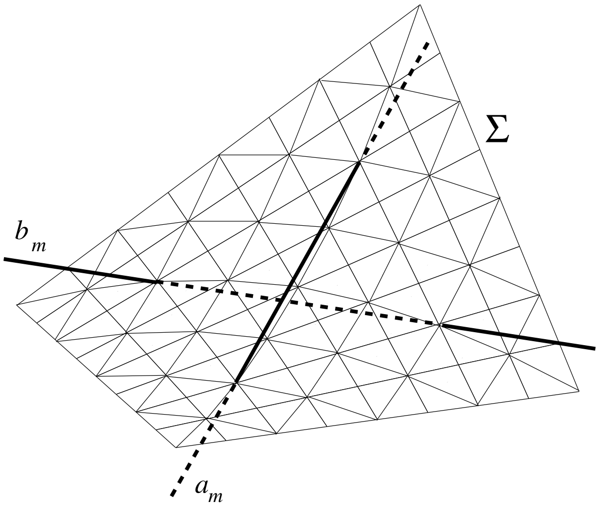

Figure 2.4: Illustration to the proof of the disjointness of A_m and B_m: the segment a_m is above b_m.

defined in the beginning of this section). The segment a_m is parallel to τ_m and lies in π_m, hence a_m is parallel to t_m, and it is easy to check that a_m is above t_m in the vertical plane π_m. We have observed that X^L is downward convex. Hence A_m^L, being a translate of X^L, is also above t_m, and thus above τ_m. Similarly, the segment b_m is parallel to s_m and is below it in the vertical plane σ_m. We have observed that X^U is upward convex. Hence B_m^U, being a translate of X^U, is also below s_m, and thus above τ_m (see Figure 2.4).

If H in the definition of X_U is large enough, then also A_m^U is above τ_m, and B_m^L is below τ_m. Since $A_m = \text{conv}(A_m^L \cup A_m^U)$ and $B_m = \text{conv}(B_m^L \cup B_m^U)$, it follows that A_m is above τ_m, and B_m is below τ_m. Hence $A_m \cap B_m = \emptyset$.

Transversal properties

We prove that each A_m and each B_m meets each l_i.

- For A_m:

  For $i = m-1$, $P_{m,1} = l_{m-1} \cap a_m$; for $i = m$, $P_{m,2} = l_m \cap a_m$.

  For each $i \neq m-1, m$: since the curve $\Sigma \cap \pi_m$ is downward convex, the point $\lambda_i \cap a_m$, which belongs to A_m^L, lies below Σ and hence below l_i. Since the projection of A_m^U on the y-axis is equal to the projection of A_m^L, there is a point of A_m^U that belongs to λ_i. Since A_m^U is high above A_m^L, this point is above l_i, and it follows that A_m meets l_i.

- For B_m:

  For $i = m-1$, $Q_{m,2} = l_{m-1} \cap b_m$; for $i = m$, $Q_{m,1} = l_m \cap b_m$.

  For each $i \neq m-1, m$: since the curve $\Sigma \cap \sigma_m$ is upward convex, the point $\lambda_i \cap b_m$, which belongs to B_m^U, lies above Σ and hence above l_i. Since the projection of B_m^L on the y-axis is equal to the projection of B_m^U, there is a point of B_m^L that belongs to λ_i. Since B_m^U is high above B_m^L, this point is below l_i, and it follows that B_m meets l_i.

Geometric permutations

Let T_m^A and T_m^B be the open halfspaces bounded by τ_m, that contain A_m and B_m respectively. Let $O_i = (0, i, 0) \in l_i$. On each l_i, choose direction such that O_i is before all the points with positive x-coordinate. It follows that for each m, on each l_i, $O_i \prec A_m$ and $O_i \prec B_m$. Note also that for $m < m'$, on each l_i we have $A_m \prec A_{m'}$, $A_m \prec B_{m'}$, $B_m \prec A_{m'}$ and $B_m \prec B_{m'}$, since the planes w_m separate such pairs of sets. However, the order of A_m and B_m on l_i depends on i:

- For $i = m-1$: on l_{m-1}, $O_{m-1} \prec P_{m,1} \prec Q_{m,2}$, and thus $A_m \prec B_m$. Note that $O_{m-1} \in T_m^A$.
- For $i = m$: on l_m, $O_m \prec Q_{m,1} \prec P_{m,2}$, and thus $B_m \prec A_m$. Note that $O_m \in T_m^B$.
- For any i: we have observed that $O_{m-1} \in T_m^A$ and $O_m \in T_m^B$. It follows that $O_i \in T_m^A$ for $i \leq m-1$, and $O_i \in T_m^B$ for $i \geq m$. Since both A_m and B_m meet each l_i after O_i, we have $A_m \prec B_m$ on l_i for $i \leq m-1$ and $B_m \prec A_m$ on l_i for $i \geq m$.

We obtain the following geometric permutations for $\mathcal{F}$:

$$\begin{aligned}
l_0 &: (A_1, B_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n)\\
l_1 &: (B_1, A_1, A_2, B_2, A_3, B_3, \ldots, A_n, B_n)\\
l_2 &: (B_1, A_1, B_2, A_2, A_3, B_3, \ldots, A_n, B_n)\\
l_3 &: (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, A_n, B_n)\\
&\ldots\\
l_n &: (B_1, A_1, B_2, A_2, B_3, A_3, \ldots, B_n, A_n).
\end{aligned}$$

Thus $\mathcal{F}$ is a family of $2n$ disjoint translates of the convex set X that has the $n+1$ geometric permutations listed above (they are distinct if $n > 1$).

2.2 Proof of Theorem 2.2

The set $X = X(n)$ that has been constructed in Section 2.1, depends on n. Here we explain, omitting some details, how to construct a set that satisfies Theorem 2.1 for all values of $n \in \mathbb{N}$.

Recall the constuction of $X(n)$. It is easy to see that it is possible to modify the construction so that λ_i is the plane $y = i\epsilon$ for some positive constant $\epsilon \leq 1$; $l_i = \lambda_i \cap \Sigma$; all the segments a_m and b_m are still parallel to the planes $y = x$ and $y = -x$ respectively, but have x- and y-lengths $n\epsilon$ (then the x- and y-lengths of X are $n^2\epsilon$). Choosing $\epsilon \leq 1/n^2$, it is possible to modify the construction so that the planes $x = 5(2m-1)$ and $x = 5(2m-1) + \epsilon$ play the roles of u_m and u'_m (respectively) in the definition of the points $P_{m,j}$ and $Q_{m,j}$, and the planes $x = 10m$ play the role of w_m in the proof of the disjointness of the translates (note that in this case the x- and y-lengths of X are less than 1). Once this is done, it is possible to "squeeze" the construction (applying the transformation $(x, y, z) \mapsto (x, y, \delta z)$ for a constant $0 < \delta \leq 1$) so that the slopes of all a_m's and b_m's will be positive but less than a constant α.

Using these observations, we construct a set Y that satisfies Theorem 2.1 for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, construct *modified* $X^L(n)$ and $X^U(n)$ so that:

1. The x- and y-lengths of $X^L(n)$ and $X^U(n)$ are $1/2^n$;

2. The slopes of all $a_m(n)$'s and $b_m(n)$'s are positive but less than the slopes of all $a_m(n-1)$'s and $b_m(n-1)$'s, and less than $1/2^n$.

Append $X^L(n)$'s ($X^U(n)$'s) in order to obtain a polygonal line Y^L (Y^U) in the way similar to the joining of the segments a_m (b_m) in the construction of X^L (X^U). That is, let $Y^L = \bigcup_{n=1}^{\infty} \tilde{X}^L(n)$ and $Y^U = \bigcup_{n=1}^{\infty} \tilde{X}^U(n)$ where $\tilde{X}^L(n)$ ($\tilde{X}^U(n)$) is a translate of *modified* $X^L(n)$ ($X^U(n)$), and the lowest point of $\tilde{X}^L(n)$ coincides with the highest point of $\tilde{X}^L(n+1)$ (the highest point of $\tilde{X}^U(n)$ coincides with the lowest point of $\tilde{X}^U(n+1)$). Because of the conditions 1 and 2 above, the sequences of the lowest points of $\tilde{X}^L(n)$ and of the highest points of $\tilde{X}^U(n)$ converge, the polygonal lines Y^L and Y^U are, respectively, downward and upward convex, and they have x- and y-lengths 1 ($= \sum_{n \in \mathbb{N}} 1/2^n$), and finite z-lengths. It remains to put Y^U high above Y^L (so that they have the same x- and y-projections, say, $[0, 1]$), and to define $Y = \operatorname{conv}(Y^L \cup Y^U)$. The set Y looks similar to the set from Figure 2, but the bold polygonal lines consist of an infinite number of segments.

For each natural n, it is possible to place $2n$ translates of Y so that the segments of Y^L (Y^U) that correspond to $a_m(n)$'s ($b_m(n)$'s) coincide with these segments in $2n$ translates of the *modified* $X(n)$.

They are disjoint since the planes $x = 10m$ separate pairs of translates from each other, and two translates forming a pair are disjoint since Y^L is and Y^U are, respectively, downward and upward convex polygonal lines.

They also admit the $n + 1$ geometric permutations mentioned in the end of Section 2.1. This concludes the proof of Theorem 2.2.

2.3 Concluding remarks

We have proved that the maximal number of geometric permutations for families of n translates in $\mathbb{R}^3$ is $\Omega(n)$. As was mentioned earlier, it is also $O(n^2)$ [21]. A natural question is: what is the exact order of this number?

Obviously, for $d \geq 3$, the maximal number of geometric permutations for such families in $\mathbb{R}^d$ is also $\Omega(n)$. It seems plausible that this bound can be improved for $d > 3$.

Chapter 3

Forbidden families of geometric permutations in $\mathbb{R}^d$

In this chapter[1] we deal with the following aspect of geometric permutations. Let $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ be some family of permutations on n sets (we assume that all of the permutations are distinct, and, furthermore, no two of them are reverses of each other). It can be asked whether there is a family $\mathcal{F}$ that admits all of them as its geometric permutations. The answer may depend on d (the dimension of the space). If there is such a family in $\mathbb{R}^d$, we say that $\mathcal{P}$ is *realizable* in $\mathbb{R}^d$, otherwise we say that it is *non-realizable* (or *forbidden*) in $\mathbb{R}^d$.

For example, the pair of permutations $\{\langle 1234\rangle, \langle 2143\rangle\}$ is forbidden in $\mathbb{R}^2$ (see Example 1 in Section 3.2.1). On the other hand, for each natural n, any pair $\{p^1, p^2\}$ of permutations on n sets is realizable in $\mathbb{R}^3$ (this is a special case of Theorem 3.1).

Our results deal with realizable and forbidden families of permutations in $\mathbb{R}^d$:

Theorem 3.1 *For each natural k, each family of k permutations is realizable in $\mathbb{R}^{2k-1}$.*

Theorem 3.2 *For each natural k, there is a family of k permutations which is non-realizable in $\mathbb{R}^{2k-2}$.*

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[1] The results from this chapter appear in the paper *Forbidden families of geometric permutations in* $\mathbb{R}^d$ [3], accepted for publication in *Discrete and Computational Geometry*; published in *Online First* issue of DCG at July 26, 2004.

It is clear that if a family of permutations is realizable in $\mathbb{R}^d$, then it is realizable in each $\mathbb{R}^{d'}$ where $d' \geq d$. Thus it follows from our theorems that each family of k permutations is realizable in $\mathbb{R}^d$ for each $d \geq 2k-1$, but there is a family of k permutations which is non-realizable in $\mathbb{R}^d$ for each $d \leq 2k-2$.

3.1 Proof of Theorem 3.1

Let $\{p^1, p^2, \ldots, p^k\}$ be a family of permutations on n sets. Take k lines $l_1, l_2, \ldots, l_k$ in general position in $\mathbb{R}^{2k-1}$ (by *general position* we mean that their affine hull is $\mathbb{R}^{2k-1}$). For each $j \in \{1, 2, \ldots, k\}$, put n points $P_{j1}, P_{j2}, \ldots, P_{jn}$ on l_j, according to the permutation p^j. For each $i \in \{1, 2, \ldots, n\}$, define $S_i = \mathrm{conv}(\{P_{1i}, P_{2i}, \ldots, P_{ki}\})$. Each S_i is of dimension at most $k-1$.

We prove that the sets $S_1, S_2, \ldots, S_n$ are pairwise disjoint: suppose $S_x \cap S_y \neq \emptyset$ with $x \neq y$. Let τ be the minimal flat containing S_x and S_y. Since S_x and S_y intersect, the dimension of τ is at most $2k-2$. Then for each $j \in \{1, 2, \ldots, k\}$, the points P_{jx} and P_{jy}, and therefore the line l_j, belong to τ. Thus all the lines $l_1, l_2, \ldots, l_k$ lie in τ, contradicting their being in general position.

Thus $\{S_1, S_2, \ldots, S_n\}$ is a family of pairwise disjoint convex sets in $\mathbb{R}^{2k-1}$, and it has $p^1, p^2, \ldots, p^k$ (induced by $l_1, l_2, \ldots, l_k$ respectively) as geometric permutations.

3.2 Proof of Theorem 3.2

We shall prove the theorem using separating hyperplanes. We shall use the following notation. For two disjoint convex sets A_x and A_y in $\mathbb{R}^d$, we denote by $H^{(xy)}$ a hyperplane ($(d-1)$-flat) that strictly separates A_x from A_y; by $H_x^{(xy)}$ the open halfspace bounded by $H^{(xy)}$ that contains A_x, and by $H_y^{(xy)}$ the open halfspace bounded by $H^{(xy)}$ that contains A_y.

3.2.1 Examples of forbidden families of permutations in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$

In this section we provide three examples of forbidden families of permutations. Examples 1 and 3 illustrate the general idea used in the proof of

Theorem 3.2; Example 2 illustrates a slightly different method, applied to $\mathbb{R}^3$.

Example 1 The pair of permutations
$\{p_1 = \langle 1\,2\;\;3\,4\rangle,$
$p_2 = \langle 2\,1\;\;4\,3\rangle\}$
is forbidden in $\mathbb{R}^2$.

Remark This example appeared already in early papers on geometric permutations [17, 19].

Proof Suppose that this pair is realizable in $\mathbb{R}^2$ with $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ and transversal lines l_1, l_2 inducing permutations p^1, p^2 respectively. Since parallel transversals clearly induce the same permutation, l_1 and l_2 intersect in a point O. Note that for each possible position of O on the transversals relative to the members of $\mathcal{F}$, there exist A_x and A_y in $\mathcal{F}$ so that $l_1\colon (O * x * y)$ and $l_2\colon (O * y * x)$, and this contradicts the disjointness of the sets:

(the bar | denotes the position of O on the transversals; * is dropped)

- $l_1\colon (|1|2|34), l_2\colon (|2|1|43)$: take $x = 3$, $y = 4$.
- $l_1\colon (12|3|4|), l_2\colon (21|4|3|)$: take $x = 2$, $y = 1$.
- $l_1\colon (|1|234), l_2\colon (214|3|)$: take $x = 2$, $y = 4$.
- $l_1\colon (123|4|), l_2\colon (|2|143)$: take $x = 3$, $y = 1$.

Example 2 The triple of permutations
$\{p_1 = \langle 1\,2\,3\;\;4\,5\,6\rangle,$
$p_2 = \langle 4\,5\,6\;\;1\,2\,3\rangle,$
$p_3 = \langle 2\,4\,6\;\;1\,3\,5\rangle\}$
is forbidden in $\mathbb{R}^3$.

Proof Suppose that this triple is realizable in $\mathbb{R}^3$ with $\mathcal{F} = \{A_1, A_2, \ldots, A_6\}$ and transversal lines l_1, l_2, l_3 inducing permutations p^1, p^2, p^3 respectively. Using standard arguments (we mention them later in Section 3.2.3), it is possible to assume that no two lines among l_1, l_2 and l_3 intersect, and that there is no plane parallel to all of them. Then there exists a line m which is parallel to l_3 and intersects both l_1 and l_2 – in points O_1 and O_2 respectively (such a line m exists since the plane τ that contains l_1 and is parallel to l_3, intersects l_2 in a point; denote this point by O_2; m is the line parallel to l_3 that contains O_2). Choose a direction for m, and the same direction for l_3, so that $m\colon (O_1 \prec O_2)$.

Suppose that there exist $A_x, A_y \in \mathcal{F}$ so that $l_1 : (O_1 * A_x * A_y)$ and $l_2\colon (O_2 * A_y * A_x)$. This implies $O_1 \in H_x^{(xy)}$ and $O_2 \in H_y^{(xy)}$, hence

$m : (H_x^{(xy)} \prec H_y^{(xy)})$, and thus also $l_3 : (H_x^{(xy)} \prec H_y^{(xy)})$. However, l_3 is a transversal of $\mathcal{F}$, hence $l_3\colon (A_x \prec A_y)$.

Note that for each possible position of O_1 and O_2 on the transversals relative to the members of $\mathcal{F}$, we can choose two pairs of members of $\mathcal{F}$ so that the previous observation contradicts the actual permutation p^3:

(the bar $|$ denotes the position of O_j on the transversal l_j):

- $l_1\colon (|1|23456)$ and $l_2\colon (|4|5|6123)$.
  $l_1\colon (|26)$ and $l_2\colon (|62)$ imply $l_3\colon (2 \prec 6)$; $l_1\colon (|36)$ and $l_2\colon (|63)$ imply $l_3\colon (3 \prec 6)$.
  A contradiction to $l_3 : (2 * 6 * 3)$.
- $l_1\colon (12|3456)$ and $l_2\colon (|4|5|6123)$.
  $l_1\colon (|21)$ and $l_2\colon (|12)$ imply $l_3\colon (2 \prec 1)$; $l_1\colon (|36)$ and $l_2\colon (|63)$ imply $l_3\colon (3 \prec 6)$.
  A contradiction to $l_3 : (2 * 6 * 1 * 3)$.
- $l_1\colon (|1|2|3|456)$ and $l_2\colon (456|1|2|3|)$.
  $l_1\colon (|46)$ and $l_2\colon (|64)$ imply $l_3\colon (4 \prec 6)$; $l_1\colon (|56)$ and $l_2\colon (|65)$ imply $l_3\colon (5 \prec 6)$.
  A contradiction to $l_3 : (4 * 6 * 5)$.
- $l_1\colon (123|4|5|6|)$ and $l_2\colon (|4|5|6|123)$.
  $l_1\colon (|21)$ and $l_2\colon (|12)$ imply $l_3\colon (2 \prec 1)$; $l_1\colon (|31)$ and $l_2\colon (|13)$ imply $l_3\colon (3 \prec 1)$.
  A contradiction to $l_3 : (2 * 1 * 3)$.
- $l_1\colon (1234|56)$ and $l_2\colon (4561|2|3|)$.
  $l_1\colon (|41)$ and $l_2\colon (|14)$ imply $l_3\colon (4 \prec 1)$; $l_1\colon (|56)$ and $l_2\colon (|65)$ imply $l_3\colon (5 \prec 6)$.
  A contradiction to $l_3 : (4 * 6 * 1 * 5)$.
- $l_1\colon (12345|6|)$ and $l_2\colon (4561|2|3|)$.
  $l_1\colon (|41)$ and $l_2\colon (|14)$ imply $l_3\colon (4 \prec 1)$; $l_1\colon (|51)$ and $l_2\colon (|15)$ imply $l_3\colon (5 \prec 1)$.
  A contradiction to $l_3 : (4 * 1 * 5)$.

Remark: In this forbidden triple, replacing p^3 by one of the seven permutations obtained from it by rearranging some of the pairs $\{2,4\}$, $\{1,6\}$, $\{3,5\}$ (for example $\langle 421635\rangle$), also gives a forbidden triple. This can be proved using the same method.

Example 3 The triple of permutations
$\{p_1 = \langle 1\,2\,3\ \ 4\,5\,6\ \ 7\,8\,9\rangle,$
$p_2 = \langle 3\,1\,2\ \ 5\,6\,4\ \ 9\,7\,8\rangle,$
$p_3 = \langle 2\,3\,1\ \ 6\,4\,5\ \ 8\,9\,7\rangle\}$
is forbidden in $\mathbb{R}^4$.

Proof: Suppose that this triple is realizable in $\mathbb{R}^4$ with $\mathcal{F} = \{A_1, A_2, \ldots, A_9\}$ and transversal lines l_1, l_2, l_3 inducing permutations p^1, p^2, p^3 respectively. It is possible to assume that there is a line s that intersects each of l_1, l_2, l_3 (this follows from Lemma 3.4 to be proved later). For $i \in \{1,2,3\}$, choose a point $O_i \in l_i \cap s$. It is also possible to assume that $O_3 \in \mathrm{conv}(\{O_1, O_2\})$: it is easy to see, using the symmetry of the permutations, that the two other possibilities can be obtained from this by relabeling the sets. Note that for each possible position of O_1, O_2, O_3 on the transversals relative to the members of $\mathcal{F}$, there exist A_x and A_y in $\mathcal{F}$ so that $l_1\colon (O_1 * x * y)$ and $l_2\colon (O_2 * x * y)$, but $l_3\colon (O_3 * y * x)$:

- $l_1\colon (|1|2|3|4|5|6|789), l_2\colon (|3|1|2|5|6|4|9|78), l_3\colon (|2|3|1|6|4|5|897)$: take $x = 7$, $y = 8$.
- $l_1\colon (12|3|4|5|6|7|8|9|), l_2\colon (312|5|6|4|9|7|8|), l_3\colon (231|6|4|5|8|9|7|)$: take $x = 2$, $y = 1$.
- $l_1\colon (|1|2|3|4|5|6789), l_2\colon (|3|1|2|5|64978), l_3\colon (2316458|9|7|)$: take $x = 6$, $y = 8$.
- $l_1\colon (12345|6|7|8|9|), l_2\colon (3125|6|4|9|7|8|), l_3\colon (|2|3|1645897)$: take $x = 5$, $y = 1$.
- $l_1\colon (|1|2|3|456789), l_2\colon (|3|1|2|5|6|4978), l_3\colon (2316458|9|7|)$: take $x = 4$, $y = 8$.
- $l_1\colon (1234|5|6|7|8|9|), l_2\colon (312564|9|7|8|), l_3\colon (|2|3|1645897)$: take $x = 4$, $y = 1$.
- $l_1\colon (|1|2|3|456789), l_2\colon (312564|9|7|8|), l_3\colon (231645|8|9|7|)$: take $x = 4$, $y = 5$.
- $l_1\colon (|1|2|3|456789), l_2\colon (312564|9|7|8|), l_3\colon (|2|3|1|645897)$: take $x = 4$, $y = 6$.
- $l_1\colon (|1|2|3|4|5|6|789), l_2\colon (31256497|8|), l_3\colon (|2|3|1|6|4|5|8|97)$: take $x = 7$, $y = 9$.
- $l_1\colon (|1|23456789), l_2\colon (312|5|6|4|9|7|8|), l_3\colon (23|1|6|4|5|8|9|7|)$: take $x = 2$, $y = 3$.
- $l_1\colon (123456|7|8|9|), l_2\colon (|3|1|2|5|64978), l_3\colon (23164|5|8|9|7|)$: take $x = 6$, $y = 4$.
- $l_1\colon (12345|6|7|8|9|), l_2\colon (|3|1|2|564978), l_3\colon (|2|3|1|6|45897)$: take $x = 5$, $y = 4$.

This contradicts the disjointness of the members of $\mathcal{F}$: $O_1, O_2 \in H_x^{(xy)}$, and $O_3 \in H_y^{(xy)}$. However, also $O_3 \in H_x^{(xy)}$ since $O_3 \in \operatorname{conv}(\{O_1, O_2\})$. Thus, O_3 belongs both to $H_x^{(xy)}$ and to $H_y^{(xy)}$, a contradiction.

3.2.2 A $(k-2)$-flat that intersects all the transversals

We prove two lemmas that imply the existence of a $(k-2)$-flat that intersects a transversal line for each of the k geometric permutation.

Definition A family $\mathcal{L}$ of s-flats in $\mathbb{R}^d$ is an *open family* if for any $L \in \mathcal{L}$, there are $s+1$ open balls $B_1, B_2, \ldots, B_{s+1}$ so that L intersects each of them, and any s-flat that intersects all these balls belongs to $\mathcal{L}$.
Remark This definition implies that for each member of an open family, a small perturbation results in another member of the family. For $s = 0$, an open family (of points) is just an open set in the usual sense.

Lemma 3.3 *Let $k \in \mathbb{N}$. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$ be open families of lines, and P a point, in $\mathbb{R}^{2k-1}$. Then there exist lines $l_i \in \mathcal{L}_i$ and a $(k-1)$-flat S so that $P \in S$ and for each $i \in \{1, 2, \ldots, k\}$, l_i intersects S.*

Lemma 3.4 *Let $k \in \mathbb{N}$, $k > 1$. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k$ be open families of lines in $\mathbb{R}^{2k-2}$. Then there exist lines $l_i \in \mathcal{L}_i$, and a $(k-2)$-flat S so that for each $i \in \{1, 2, \ldots, k\}$, l_i intersects S.*

Proof of Lemma 3.3

For $k = 1$ the statement is obvious.

Suppose the Lemma holds for $k - 1$.

For $1 \leq i \leq k$, it is possible to choose $l_i \in \mathcal{L}_i$ so that: $A = \text{aff}\{l_1, l_2, \ldots, l_{k-1}\}$ is a $(2k-3)$-flat; $B = \text{aff}\{l_k, P\}$ is a 2-flat; and $A \cap B$ is a point Q different from P, so that the line PQ intersects l_k.

For $1 \leq i \leq k-1$, let $\mathcal{L}'_i = \mathcal{L}_i \cap A$. Each $\mathcal{L}'_i$ is an open family of lines relative to A. By the induction assumption applied to $\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{k-1}$ and Q in the $(2k-3)$-flat A, there exist $l'_i \in \mathcal{L}'_i \subseteq \mathcal{L}_i$ $(1 \leq i \leq k-1)$, and a $(k-2)$-flat T that contains Q and intersects each l'_i.

Let $S = \text{aff}(T, P)$. Clearly, $P \in S$. The flat S intersects each l'_i since $T \subseteq S$, and it intersects l_k since the line PQ lies in S. Since Q is the only point in $A \cap B$, S is a $(k-1)$-flat.

Thus, S and the lines $l'_1, l'_2, \ldots, l'_{k-1}, l_k$ satisfy the conclusion of the Lemma.

Proof of Lemma 3.4

For $1 \leq i \leq k$, it is possible to choose $l_i \in \mathcal{L}_i$ so that $C = \text{aff}\{l_1, l_2, \ldots, l_{k-1}\}$ is a $(2k-3)$-flat, and $l_k \cap C$ is a point P.

For $1 \leq i \leq k-1$, let $\mathcal{L}'_i = \mathcal{L}_i \cap C$. Each $\mathcal{L}'_i$ is an open family of lines in C. By Lemma 3.3 applied to $\mathcal{L}'_1, \mathcal{L}'_2, \ldots, \mathcal{L}'_{k-1}$ and P in the $(2k-3)$-flat C, there exist $l'_i \in \mathcal{L}'_i$ $(1 \leq i \leq k-1)$, and a $(k-2)$-flat S that intersects each l'_i and contains P (that belongs to l_k).

Thus, S and the lines $l'_1, l'_2, \ldots, l'_{k-1}, l_k$ satisfy the conclusion of the Lemma.

3.2.3 Idea of the proof of Theorem 3.2

Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a family of disjoint convex sets in $\mathbb{R}^{2k-2}$ that has permutations $p^1, p^2, \ldots, p^k$. After a slight expansion of the members of $\mathcal{F}$, for each geometric permutation there is a transversal line that intersects all the members of $\mathcal{F}$ in interior points. Then, for each i, the family of all the transversal lines that induce p^i contains an open family of lines. Hence, by Lemma 3.4, it is possible to choose transversals $l_1, l_2, \ldots, l_k$ (inducing

$p^1, p^2, \ldots, p^k$ respectively) so that there is a $(k-2)$-dimensional flat S that intersects each of these transversals.

For each $j \in \{1, 2, \ldots, k\}$, let $O_i \in l_i \cap S$. These are k points in a $(k-2)$-flat, thus **by Radon's Theorem** [22] they can be partitioned into two non-empty sets whose convex hulls intersect: $\{1, 2, \ldots, k\} = K \cup L$, $K \cap L = \emptyset$, $K, L \neq \emptyset$, and $\mathrm{conv}(\{O_j : j \in K\}) \cap \mathrm{conv}(\{O_j : j \in L\}) \neq \emptyset$.

Suppose that there are two sets A_x and A_y in $\mathcal{F}$ so that for each $j \in K$, $l_j : (O_j * A_x * A_y)$, and for each $j \in L$, $l_j : (O_j * A_y * A_x)$. Then for each $j \in K$, $O_j \in H_x^{(xy)}$, and for each $j \in L$, $O_j \in H_y^{(xy)}$. Since the open halfspaces $H_x^{(xy)}$ and $H_y^{(xy)}$ are convex sets, it follows that $\mathrm{conv}(\{O_j : j \in K\}) \subseteq H_x^{(xy)}$ and $\mathrm{conv}(\{O_j : j \in L\}) \subseteq H_y^{(xy)}$. However, then each point common to $\mathrm{conv}(\{O_j : j \in K\})$ and $\mathrm{conv}(\{O_j : j \in L\})$ belongs to both $H_x^{(xy)}$ and $H_y^{(xy)}$, which is clearly impossible.

Thus, we have proved the following:

Observation 3.5 *If a family of permutations $\{p^1, p^2, \ldots, p^k\}$ for $\mathcal{F}$ is such that for each partition of $\{1, 2, \ldots, k\}$ into two disjoint non-empty sets K and L, and for each possible position of the O_j's in the p^j's relative to the members of $\mathcal{F}$, there are two sets A_x and A_y in $\mathcal{F}$ (that depend on the partition and on the position of the O_j's) so that for each $j \in K$, $l_j : (O_j * A_x * A_y)$, and for each $j \in L$, $l_j : (O_j * A_y * A_x)$ – such a family of permutations is forbidden in $\mathbb{R}^{2k-2}$.*

3.2.4 Construction of a forbidden family of permutations

We construct a family of k permutations $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ that has the property mentioned in Observation 3.5. The permutations involve $(k+1) \cdot (2^{k-1}+1)$ sets. In the first step we construct their subpermutations $\pi^1, \pi^2, \ldots, \pi^k$ which are permutations of $\{0, 1, \ldots, 2^{k-1}\}$.

Let $S_1, S_2, \ldots, S_{2^{k-1}}$ be the 2^{k-1} subsets of $\{1, 2, \ldots, k\}$ that contain 1 (numbered in some way). Define $\pi^1, \pi^2, \ldots, \pi^k$ to be permutations of $\{0, 1, \ldots, 2^{k-1}\}$ that satisfy:

In π^j: 0 is before $i \Leftrightarrow j \in S_i$.

Note that the permutations $\pi^1, \pi^2, \ldots, \pi^k$ are not defined uniquely.

After that, for each $j \in \{1, 2, \ldots, k\}$, construct a permutation p^j by duplication of π^j $k+1$ times, as follows: for $\pi^j = (\alpha_0, \alpha_1, \ldots, \alpha_{2^{k-1}})$, define

$p^j = \big((\alpha_0, 1), \ldots, (\alpha_{2^{k-1}}, 1), (\alpha_0, 2), \ldots, (\alpha_{2^{k-1}}, 2), \ldots\ldots, (\alpha_0, k+1), \ldots, (\alpha_{2^{k-1}}, k+1)\big)$. The permutations p^1, p^2, …, p^k are permutations of the members of the set $\{0, 1, \ldots, 2^{k-1}\} \times \{1, 2, \ldots, k+1\}$. For each $m \in \{1, 2, \ldots, k+1\}$, we call the subpermutation $((\alpha_0, m), \ldots, (\alpha_{2^{k-1}}, m))$ *the m-th interval of p^i*.

Example of the construction for $k = 3$: Let $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2\}$, $S_3 = \{1, 3\}$, $S_4 = \{1\}$. The permutations π^1, π^2, π^3 should be defined so that:

In π^1: $0 \prec 1$, $0 \prec 2$, $0 \prec 3$, $0 \prec 4$;

In π^2: $0 \prec 1$, $0 \prec 2$, $3 \prec 0$, $4 \prec 0$;

In π^3: $0 \prec 1$, $2 \prec 0$, $0 \prec 3$, $4 \prec 0$.

For example, take $\pi^1 = (01234)$, $\pi^2 = (34012)$, $\pi^3 = (24013)$.

Then, for these choices of π^1, π^2 and π^3,

$$p^1 = \big(\underbrace{(0,1),(1,1),(2,1),(3,1),(4,1)}_{1^{st} interval}, \underbrace{(0,2),(1,2),(2,2),(3,2),(4,2)}_{2^{nd} interval}, \ldots, \underbrace{(0,4),(1,4),(2,4),(3,4),(4,4)}_{4^{th} interval}\big),$$
$$p^2 = \big((3,1),(4,1),(0,1),(1,1),(2,1),(3,2),(4,2),(0,2),(1,2),(2,2),\ldots,(3,4),(4,4),(0,4),(1,4),(2,4)\big),$$
$$p^3 = \big((2,1),(4,1),(0,1),(1,1),(3,1),(2,2),(4,2),(0,2),(1,2),(3,2),\ldots,(2,4),(4,4),(0,4),(1,4),(3,4)\big).$$

The family of permutations $\{p^1, p^2, p^3\}$ is forbidden in $\mathbb{R}^4$.

3.2.5 Why the construction gives a forbidden family

We prove that the family of permutations $\{p^1, p^2, \ldots, p^k\}$ defined in Section 3.2.4 is forbidden in $\mathbb{R}^{2k-2}$. Suppose that there exists a family $\mathcal{F}$ of convex disjoint sets in $\mathbb{R}^{2k-2}$ that admits p^1, p^2, …, p^k as geometric permutations. Let l_1, l_2, …, l_k be transversals giving these geometric permutations, and let S be a $(k-2)$-flat that intersects each l_j, and let $O_j \in l_j \cap S$. Since each p^j consists of $(k+1)$ "intervals", there is $m \in \{1, 2, \ldots, k+1\}$ so that for each $j \in \{1, 2, \ldots, k\}$, O_j does not belong to the m-th interval of p^j. After dropping the "second component" (m), the m-th interval of p^j is identical to π^j, and O_j is either before or after all of its sets.

Let $K \cup L$ be a partition of $\{1, 2, \ldots, k\}$ into two disjoint non-empty sets. Define $M = \{j \in \{1, 2, \ldots, k\} : O_j \text{ is before } \pi^j\}$ and $N = \{j \in \{1, 2, \ldots, k\} :$ $O_j \text{ is after } \pi^j\}$. Define $K' = (K \cap M) \cup (L \cap N)$. Note that K' is a subset of $\{1, 2, \ldots, , k\}$, and it is possible to assume that $1 \in K'$ (otherwise we interchange K and L). Hence $K' = S_a$ for some $a \in \{1, 2, \ldots, 2^{k-1}\}$.

Consider four cases:

- If $j \in K \cap M$, then $j \in K' = S_a$, hence $l_j : (O_j \prec A_0 \prec A_a)$.
- If $j \in K \cap N$, then $j \notin K' = S_a$, hence $l_j : (A_a \prec A_0 \prec O_j)$.
- If $j \in L \cap M$, then $j \notin K' = S_a$, hence $l_j : (O_j \prec A_a \prec A_0)$.
- If $j \in L \cap N$, then $j \in K' = S_a$, hence $l_j : (A_0 \prec A_a \prec O_j)$.

In each case, for each $j \in K$, $l_j : (O_j * A_0 * A_a)$, and for each $j \in L$, $l_j : (O_j * A_a * A_0)$. Then Observation 3.5 implies that the family of permutations is forbidden.

3.3 Bounds on the minimal number of sets in a forbidden family

By Theorems 3.1 and 3.2, for each natural d, each family of $\lceil d/2 \rceil$ permutations is realizable in $\mathbb{R}^d$, but there is a forbidden family of $\lceil d/2 \rceil + 1$ permutations. What is the minimal number of sets that must be involved in such a forbidden family? Denote this minimal number by φ_d. By our proof, $\varphi_d \leq (\lceil d/2 \rceil + 2) \cdot (2^{\lceil d/2 \rceil} + 1)$. This gives $\varphi_2 \leq 9$ and $\varphi_3, \varphi_4 \leq 20$, whereas, by Examples from Section 3.2.1, $\varphi_2 \leq 4$, $\varphi_3 \leq 6$ and $\varphi_4 \leq 9$.

On the other hand, for each natural d, there is a family of $d + 1$ disjoint convex sets in $\mathbb{R}^d$ that have all possible $(d+1)!/2$ geometric permutations [18]. It follows that $\varphi_d \geq d + 2$. Thus, $\varphi_2 = 4$, $5 \leq \varphi_3 \leq 6$, $6 \leq \varphi_4 \leq 9$. It seems that the exponential upper bound for φ_d can be improved substantially. It is clear that $d \leq d'$ implies $\varphi_d \leq \varphi_{d'}$. However, we do not even know whether φ_d is strictly monotone as a function of d.

Chapter 4

Geometric permutations in the plane

In this chapter we study realizable and forbidden families of permutations in the plane $\mathbb{R}^2$. For $|\mathcal{P}| = 2$, i.e. for families of *two* permutations, we give a complete characterization of realizability. For $|\mathcal{P}| > 2$ we give some necessary conditions for the realizability, and provide examples which show that the conditions are not sufficient.

4.1 Notations and preliminary results

Realization of a family of permutations; Regular realization.

A *realization* of $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ in $\mathbb{R}^d$ is a pair $(\mathcal{F}, \mathcal{L})$ where $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ is a family of disjoint convex sets in $\mathbb{R}^d$, $\mathcal{L} = \{l_1, l_2, \ldots, l_k\}$ is a family of transversal lines of $\mathcal{F}$ so that l_j induces the permutation p^j on the members of $\mathcal{F}$. (Thus a family of permutations $\mathcal{P}$ is realizable in $\mathbb{R}^d$ if and only if there is a realization of $\mathcal{P}$ in $\mathbb{R}^d$.)

Observation 4.1 *Let $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ be a realizable family of permutations on n sets. Then there is a realization of $\mathcal{P}$ in which the sets are convex polygons whose vertices lie on the transversal lines, and none of the vertices coincides with an intersection point of transversals.*

Proof Let $(\mathcal{F} = \{A_1, A_2, \ldots, A_n\}, \mathcal{L} = \{l_1, l_2, \ldots, l_k\})$ be a realization of $\mathcal{P}$. For each $j = 1, 2, \ldots, k$, $i = 1, 2, \ldots, n$, choose a point $P_{ji} \in l_j \cap A_i$. It is possible to assume that no P_{ji} is an intersection point of transversal lines (a slight expansion of the members of $\mathcal{F}$ may be needed). For each $i = 1, 2, \ldots, n$, define B_i to be the polygon $\mathrm{conv}(P_{1i}, P_{2i}, \ldots, P_{ki})$. Then the family $(\mathcal{F}' = \{B_1, B_2, \ldots, B_n\}, \mathcal{L})$ is the required realization. ■

A realization described in Observation 4.1 will be called *regular*. For $\mathcal{P} = \{p, q\}$, a family of two permutations, the sets in a regular realization are segments with endpoints on the transversal lines (and none of the endpoints is the point of the intersection of the transversals).

Quadrants. Let l_1 and l_2 be two directed nonparallel lines in $\mathbb{R}^2$, and let O be the point of their intersection. For $j = 1, 2$, let l_j^+ (l_j^-) denote the positive (negative) part of l_j. Denote the four closed quadrants formed by l_1 and l_2 as follows: I bounded by l_1^+ and l_2^+, II bounded by l_1^- and l_2^+, III bounded by l_1^- and l_2^-, IV bounded by l_1^+ and l_2^-.

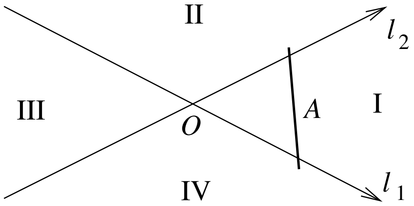

Figure 4.1: Four quadrants defined by l_1 and l_2. The set A crosses quadrant I.

We say that a set A *crosses* one of the quadrants defined by l_1, l_2 if it intersects both the rays that bound the quadrant, but does not contain the point O (see Figure 4.1). It is clear that if l_1 and l_2 intersect A, then either A contains O, or A crosses just one of the four quadrants. Therefore each segment in a regular realization of a family of two permutations crosses just one quadrant.

The graph of a permutation. Sometimes, in order to illustrate a discussion on a permutation $p = (p_1, p_2, \ldots, p_n)$, we shall use the *graph of* p, that is the set of points $\{(1, p_1), (2, p_2), \ldots, (n, p_n)\}$ in the plane. See Figure 4.2.

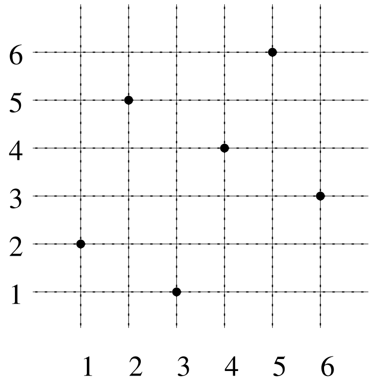

Figure 4.2: The graph of the permutation $p = (251463)$

4.2 A Helly-type result on decomposable sequences

Claim 4.2 *The family of permutations* $\mathcal{P} = \{p = \langle 1234\rangle, q = \langle 2143\rangle\}$ *is forbidden in* $\mathbb{R}^2$.

This fact was noted in one of the earliest papers on geometric permutations [19]. A detailed proof can be found in [3]. It can be also proved using Observation 4.14 from Section 4.6

Now we define a property of a permutation to be decomposable to an ascending and a descending subpermutations. The definition is valid for a wider class of objects: for finite one to one sequences of real numbers.

Let $a = (a_1, a_2, \ldots, a_n)$ be a one to one real function (sequence) defined on $[n] = \{1, 2, \ldots, n\}$. For $A \subseteq [n]$, let $a|_A$ be the the restriction of a to A. For example, if $n = 5$, $a = (2, 6, 1, 4, 8)$ and $A = \{2, 3, 5\}$, then $a|_A = (6, 1, 8)$. (Note that the members of A refer to the places in p, and not to their values.)

Definition. A one to one sequence $a = (a_1, a_2, \ldots, a_n)$ is *decomposable* if there exists a partition of $[n]$, $K \cup M = [n]$, $K \cap M = \emptyset$, so that for $i, j \in K$, $i < j \Rightarrow a_i < a_j$, and for $i, j \in M$, $i < j \Rightarrow a_i > a_j$ (in the other words: $a|_K$ is an ascending subsequence, and $a|_M$ is a descending subsequence).

Note that a permutation is decomposable if and only if its reverse is decomposable (K and M interchange roles), and thus the property of being decomposable is well defined for undirected permutations.

Observation 4.3 *For* $n = 1, 2, 3$*, each permutation of* $[n]$ *is decomposable.*

Observation 4.4 *For $n = 4$, there are just two non-decomposable permutations of* $[4]$*:* (2143) *and* (3412).

Figure 4.3 shows the graphs of these permutations.

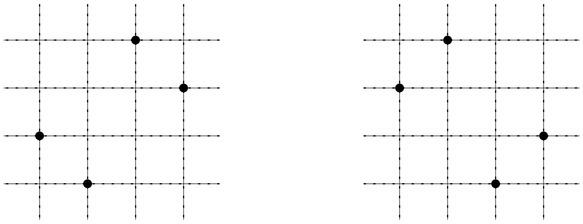

Figure 4.3: The graphs of the only non-decomposable permutations of $\{1, 2, 3, 4\}$.

Observation 4.5 *If $A = \{i, j, k, l\}$ with $i < j < k < l$, then $a|_A$ is non-decomposable if and only if $a_j < a_i < a_l < a_k$ or $a_k < a_l < a_i < a_j$.*

Observations 4.3 and 4.4 can be proved by checking all the cases. Observation 4.5 follows directly from Observation 4.4. Note that the only non-decomposable permutations of [4] are precisely the representatives of the undirected permutation $\langle 2143 \rangle$ mentioned in Claim 4.2.

The main result of this section is the following Theorem which claims that a sequence a is decomposable if and only if no restriction of a to 4 members has one of the types mentioned in Observation 4.5

Theorem 4.6 *A one to one sequence $a = (a_1, a_2, \ldots, a_n)$ is decomposable if and only if for each $A \subseteq [n]$, $|A| = 4$, the restriction $a|_A$ is decomposable.*

Proof. The necessity is clear. We shall prove the sufficiency.

For $n = 1, 2, 3$, the claim holds in a trivial way.

For $n = 4$ the claim is also trivial.

It remains to prove the sufficiency for $n > 4$. We do it by induction on n.

Let $a = (a_1, a_2, \ldots, a_n)$ be a one to one sequence defined on $[n]$ $(n > 4)$ such that for each $A \subseteq [n]$, $|A| = 4$, the restriction $a|_A$ is decomposable. It is clear that for each $B \subseteq [n-1]$, $|B| = 4$, the restriction $a|_B$ is decomposable, and thus, by the induction hypothesis, $a|_{[n-1]}$ is decomposable. Let (K, M)

be a corresponding partition of $[n-1]$: $K \cup M = [n-1]$, $K \cap M = \emptyset$, so that for $i, j \in K$, $i < j \Rightarrow a_i < a_j$, and for $i, j \in M$, $i < j \Rightarrow a_i > a_j$.

If $K = \emptyset$, set $K' = \{n\}$. Then $(a|_{K'}, a|_M)$ is a decomposition of a.

Similarly, if $M = \emptyset$, set $M' = \{n\}$. Then $(a|_K, a|_{M'})$ is a decomposition of a.

From now, assume $K \neq \emptyset$, $M \neq \emptyset$. Denote $\kappa = \max\{i : i \in K\}$, $\mu = \max\{i : i \in M\}$. It is clear that $\kappa < n$, $\mu < n$. It follows from the definitions of K and M that $a_\kappa = \max\{a_i : i \in K\}$, $a_\mu = \min\{a_i : i \in M\}$.

If $a_n > a_\kappa$, set $K' = K \cup \{n\}$. Then $(a|_{K'}, a|_M)$ is a decomposition of a.

Similarly, if $a_n < a_\mu$, set $M' = M \cup \{n\}$. Then $(a|_K, a|_{M'})$ is a decomposition of a.

It remains to check the case $a_\mu < a_n < a_\kappa$.

There is no loss of generality in assuming that $\kappa < \mu$ since otherwise we replace $a = (a_1, a_2, \ldots, a_n)$ by $-a = (-a_1, -a_2, \ldots, -a_n)$ and exchange the roles of K and M; it is obvious that a is decomposable if and only if $-a$ is decomposable, and that $a|_A$ a is decomposable if and only if $(-a)|_A$ is decomposable.

To summarize, from here on we assume $\kappa < \mu < n$, and $a_\mu < a_n < a_\kappa$.

There is no $i \in [n]$ such that $\kappa < i < \mu$ and $a_i > a_\kappa$ since otherwise we have $\kappa < i < \mu < n$ and $a_\mu < a_n < a_\kappa < a_i$ and thus $a|_{\{\kappa,i,\mu,n\}}$ is non-decomposable.

Similarly, there is no $i \in [n]$ such that $i < \kappa$ and $a_n < a_i < a_\kappa$ since otherwise we have $i < \kappa < \mu < n$ and $a_\mu < a_n < a_i < a_\kappa$ and thus $a|_{\{\mu,n,i,\kappa\}}$ is non-decomposable.

Besides, there there is no $i \in [n] \setminus \{n\}$ such that $i > \mu$, and there is no $i \in [n]$ such that $\kappa < i < \mu$ and $a_i < a_\mu$: these facts follow from the definitions of κ and μ, and from the assumption $\kappa < \mu$.

We summarize these observations in Figure 4.4. It presents the points that correspond to a_κ, a_μ and a_n on the graph of the permutation a. The shaded regions do not contain points.

We consider two cases.

Case 1: *There is no* $m \in M$ *such that* $m < \kappa$ *and* $a_\mu < a_m < a_n$. This means that for $m \in M \setminus \{\mu\}$, if $m < \kappa$, then $a_m > a_\kappa$, and if $\kappa < m < \mu$, then $a_\mu < a_m < a_\kappa$ (see Figure 4.5). Therefore if we define $K' = (K \setminus \{\kappa\}) \cup \{n\}$, $M' = M \cup \{\kappa\}$, then the partition (K', M') of $[n]$ gives a decomposition of a.

Case 2: *There is* $m \in M$ *such that* $m < \kappa$ *and* $a_\mu < a_m < a_n$. We note first that such m is unique. For if there are $m_1, m_2 \in M$, $m_1 \neq m_2$, that satisfy this. Assume $m_1 < m_2$. Then $a_{m_2} < a_{m_1} < a_n$ (since $m_1, m_2 \in M$).

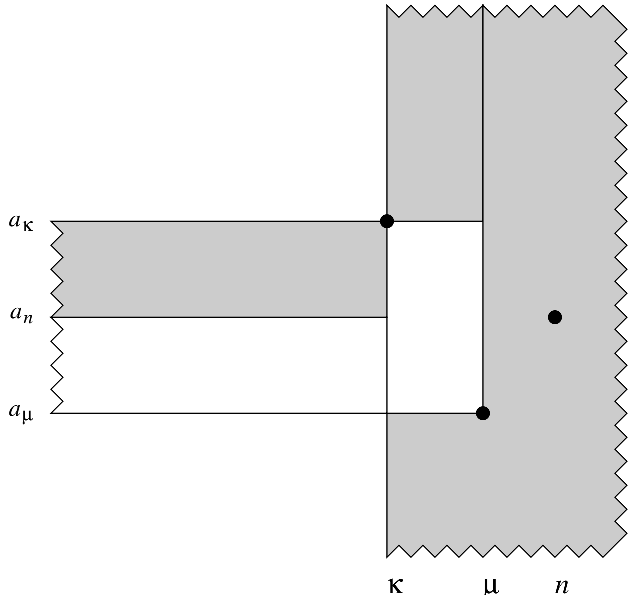

Figure 4.4: Theorem 4.6. The shaded regions do not contain points.

Now $m_1 < m_2 < \kappa < n$ with $a_{m_2} < a_{m_1} < a_n < a_\kappa$ and thus $a|_{\{m_1,m_2,\kappa,n\}}$ is non-decomposable.

There is no $i \in [n]$ such that $i < m$ and $a_m < a_i < a_n$ since otherwise $i < m < \kappa < n$ with $a_m < a_i < a_n < a_\kappa$ and thus $a|_{\{i,m,\kappa,n\}}$ is non-decomposable, and there is no $i \in [n]$ such that $m < i < \kappa$ and $a_i < a_m$ since otherwise $m < i < \kappa < n$ with $a_i < a_m < a_n < a_\kappa$ and thus $a|_{\{m,i,\kappa,n\}}$ is non-decomposable.

This means that for $i \in K$, if $i < m$ then $a_i < a_m$, and if $m < i < \kappa$ then $a_m < a_i < a_n$; and for $i \in M$, if $i < m$ then $a_i > a_\kappa$, $m < i < \kappa$ is impossible (since $m \in M$), and if $\kappa < i < \mu$ then $a_\mu < a_i < a_m$.

These observations are summarized in Figure 4.6.

It follows that if we define $K' = (K \setminus \{\kappa\}) \cup \{m\} \cup \{n\}$, $M' = (M \setminus \{m\}) \cup \{\kappa\}$, then the partition (K', M') of $[n]$ gives a decomposition of a.

Remark. Theorem 4.6 was proved by Z. Stankova in [24]. Our proof is independent (however, it is essentially identical to her proof).

Corollary. A permutation $a = (a_1, a_2, \ldots, a_n)$ of $[n]$ is decomposable if and only if for each $A \subseteq [n]$, $|A| = 4$, the restriction $a|_A$ is decomposable.

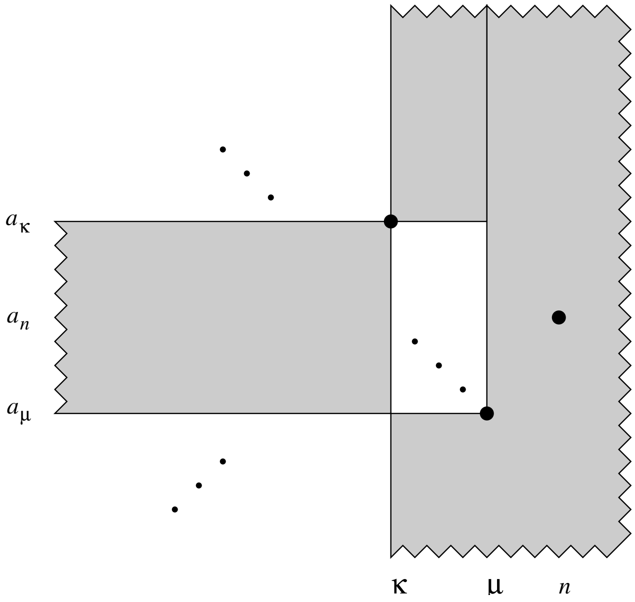

Figure 4.5: Case 1 in Theorem 4.6.

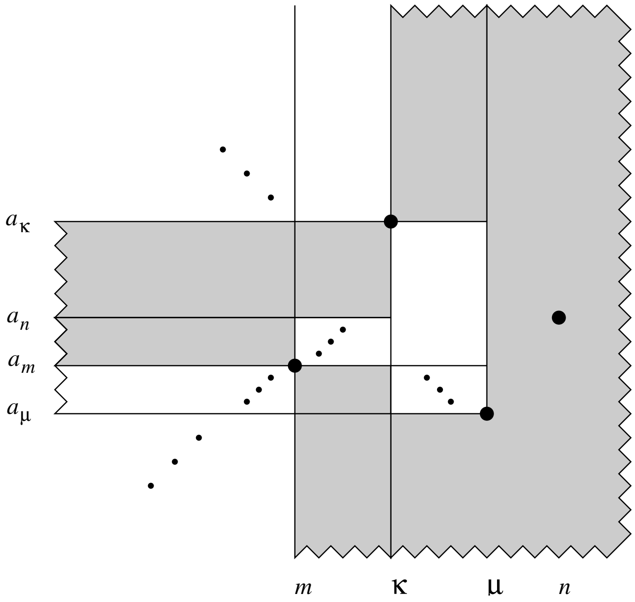

Figure 4.6: Case 2 in Theorem 4.6.

4.3 A relation between realizable pairs of geometric permutations and decomposable finite sequences

In this Section we prove another condition equivalent to realizability of a family $\mathcal{P} = \{p, q\}$ of two permutations in $\mathbb{R}^2$. Since it is possible to relable the sets so that p will be $(1, 2, 3, \dots, n)$, we shall assume this in the main theorem of this Section (Theorem 4.7). The result without this assumption will be formulated as Corollary 4.8.

Theorem 4.7 *A family* $\{p = (1, 2, \dots, n), q = (q_1, q_2, \dots, q_n)\}$ *of permutations for* $[n]$ *is realizable (as geometric permutations of some family) in* $\mathbb{R}^2$ *if and only if* q *is decomposable.*

Proof. Necessity. Let $\mathcal{P} = \{p = (1, 2, \dots, n), q = (q_1, q_2, \dots, q_n)\}$ be a family of permutations for $[n]$ realizable in $\mathbb{R}^2$. Let $(\mathcal{F} = \{A_1, A_2, \dots, A_n\}, \mathcal{L} = \{l_p, l_q\})$ be a regular realization of $\mathcal{P}$ (each A_i is a segment).

Each $A_i \in \mathcal{F}$ belongs to one of the four quadrants. Denote $K = \{i \in [n] : A_{q_i} \in \mathrm{I} \cup \mathrm{III}\}$, $M = \{i \in [n] : A_{q_i} \in \mathrm{II} \cup \mathrm{IV}\}$. It is clear that $K \cup M = [n]$, $K \cap M = \emptyset$. Since the sets are disjoint, for $i, j \in K$, $i < j \Rightarrow q_i < q_j$, and for $i, j \in M$, $i < j \Rightarrow q_i > q_j$ (see Figure 4.7). Thus q is decomposable.

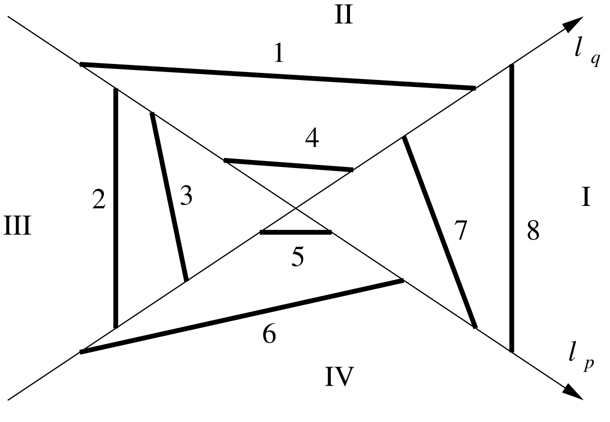

Figure 4.7: A realizable pair $\mathcal{P} = \{p = (12345678), q = (62354718)\}$. For $K = \{2, 3, 6, 8\}$, $M = \{1, 4, 5, 7\}$: $q|_K = (2378)$, $q|_M = (6541)$.

Sufficiency. Suppose q is a decomposable permutation and $[n] = K \cup M$ is a partition as appears in the definition of a decomposable sequence. For each $\alpha \in \{0.5, 1.5, \dots, n+0.5\}$, let $K_L^\alpha = \{i \in K : i < \alpha\}$, $K_R^\alpha = \{i \in K : i > \alpha\}$, $M_L^\alpha = \{i \in M : i < \alpha\}$, $M_R^\alpha = \{i \in M : i > \alpha\}$, and denote (for nonempty sets) $\lambda'_\alpha = \max\{i : i \in K_L^\alpha\}$, $\lambda''_\alpha = \max\{i : i \in M_L^\alpha\}$, $\rho'_\alpha = \min\{i : i \in K_R^\alpha\}$, $\rho''_\alpha = \min\{i : i \in M_R^\alpha\}$.

We say that α is *good* if two following conditions hold:

- $K_L^\alpha = \emptyset$, or $M_L^\alpha = \emptyset$, or $q_{\lambda'_\alpha} < q_{\lambda''_\alpha}$ (α is *good from the left*);
- $K_R^\alpha = \emptyset$, or $M_R^\alpha = \emptyset$, or $q_{\rho'_\alpha} < q_{\rho''_\alpha}$ (α is *good from the right*).

Example. Let $q = (62354718)$, $K = \{2, 3, 6, 8\}$, $M = \{1, 4, 5, 7\}$. Then $q|_K = (2378)$, $q|_M = (6541)$ (see Figure 4.8). The good from left values of α are $\alpha = 0.5$, 1.5, 2.5, 3.5, 4.5, 5.5; the good from right values of α are $\alpha = 3.5$, 4.5, 5.5, 6.5, 7.5, 8.5. Thus the good values of α are $\alpha = 3.5$ (with $q_{\lambda'} = 3$, $q_{\lambda'} = 6$, $q_{\rho'} = 7$, $q_{\rho''} = 5$); $\alpha = 4.5$ (with $q_{\lambda'} = 3$, $q_{\lambda'} = 6$, $q_{\rho'} = 7$, $q_{\rho''} = 4$); and $\alpha = 5.5$ (with $q_{\lambda'} = 3$, $q_{\lambda'} = 4$, $q_{\rho'} = 7$, $q_{\rho''} = 1$).

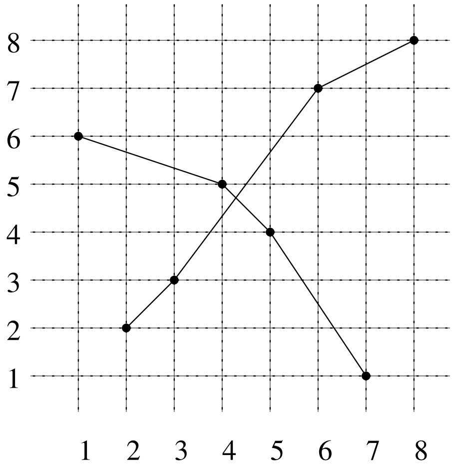

Figure 4.8: The graph of $q = (62354718)$ and the decomposition for $K = \{2, 3, 6, 8\}$, $M = \{1, 4, 5, 7\}$.

We shall prove that there is at least one good α. Trivially, $\alpha = 0.5$ is good from the left. Let $\beta = b + 0.5$ be the maximal (the rightmost) good from the left α.

If $\beta = n + 0.5$ then it is trivially good from the right and thus good.

Let $\beta < n + 0.5$, and suppose that it is not good from the right. This means that $K_R^\beta \neq \emptyset$ and $M_R^\beta \neq \emptyset$, and $q_{\rho''_\beta} > q_{\rho'_\beta}$. We claim that $\beta + 1$ is also good from the left, contradicting the choice of β.

It can be assumed without loss of generality that $b + 1 = \beta + 0.5 \in K$ (otherwise we can replace q by q', the permutation defined by $q'_i = n - q_i$. Then K and M interchange their roles, and so also λ'_α and λ''_α, and ρ'_α and ρ''_α for each α). Then $\lambda'_{\beta+1} = b + 1 = \rho'_\beta$. If $M_L^\beta = \emptyset$ then also $M_L^{\beta+1} = \emptyset$ (since $b + 1 \in K$), and thus $\beta + 1$ is good from the left. If $M_L^\beta \neq \emptyset$ then $\lambda''_{\beta+1} = \lambda''_\beta$ (since $b + 1 \in K$), therefore $q_{\lambda'_{\beta+1}} = q_{\rho'_\beta} < q_{\rho''_\beta} < q_{\lambda''_\beta} = q_{\lambda''_{\beta+1}}$ and thus $\beta + 1$ is good from the left.

Now we construct a family $\mathcal{F}$ of n disjoint segments that has geometric permutations $p = (123 \ldots n)$ and q. Let (K, M) be a decomposition of q. Choose a good α for this decomposition. Let l_p and l_q be two intersecting directed lines. Put points $P_1, P_2, \ldots, P_n$ on l_p, and $Q_1, Q_2, \ldots, Q_n$ on l_p in the orders defined by p and q, so that

1. If $i \in K_L^\alpha \cup M_R^\alpha$ then $P_{q_i} \in l_p^-$; if $i \in K_R^\alpha \cup M_L^\alpha$ then $P_{q_i} \in l_p^+$.
2. If $i \in K_L^\alpha \cup M_L^\alpha$ then $Q_{q_i} \in l_q^-$; if $i \in K_R^\alpha \cup M_R^\alpha$ then $Q_{q_i} \in l_q^+$.

It is possible to satisfy the first condition: suppose $i \in K_L^\alpha \cup M_R^\alpha$ and $j \in K_R^\alpha \cup M_L^\alpha$. If $i \in K_L$, $j \in K_R^\alpha$, or if $i \in \cup M_R^\alpha$, $j \in \cup M_L^\alpha$ then $q_i < q_j$ by definitions of K and L. If $i \in K_L^\alpha$, $j \in M_L^\alpha$ then $q_i < q_j$ since α is good from left. If $i \in M_R^\alpha$, $j \in K_R^\alpha$ then $q_i < q_j$ since α is good from right.

It is also possible to satisfy the second condition: this follows directly from the definitions of K_L^α, M_L^α, K_R^α and M_R^α.

For each $s \in [n]$, denote by A_s the segment $P_s Q_s$. The segments do not intersect for the following reason. If two segments, $P_{q_i} Q_{q_i}$ and $P_{q_j} Q_{q_j}$, with $q_i < q_j$, intersect, they belong to the same quadrant. Suppose, for example, that they belong to quadrant II (bounded by l_p^- and l_q^+). Then $i, j \in M_R^\alpha$. It follows that $l_p : (Q_{q_i} \prec Q_{q_j} \prec O)$ and $l_q : (O \prec Q_{q_j} \prec Q_{q_i})$. Therefore A_s and A_t do not intersect. Similar arguments prove the statement for the other quadrants. ■

Corollary 4.8 *A family $\{p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n)\}$ of two permutations for $[n]$ is realizable in $\mathbb{R}^2$ if and only if the members of $[n]$ can be partitioned into two subsets so that the members of one part appear in the same order in p and q, and the members of the second part appear in opposite orders in p and q.*

Corollary 4.8 generalizes Theorem 4.7 for the case when the permutation p is not necessarily $(123\ldots n)$, and it follows directly from this Theorem by relabeling the sets.

Combining Theorem 4.7 ($\{p = (123\ldots n), q\}$ is realizable in $\mathbb{R}^2$ if and only if q is decomposable) with Theorem 4.6 (q is decomposable if and only if each restriction of q to four sets is decomposable) and Observations 4.4 and 4.5 ((2143) and (3412) are the only non-decomposable permutations for $[n]$), and relabeling the sets if necessary, we obtain the following Helly-type condition of realizability of $\{p, q\}$ in $\mathbb{R}^2$:

Theorem 4.9 *A family $\{p, q\}$ of permutations for $[n]$ is forbidden in $\mathbb{R}^2$ if and only if there are $i, j, k, l \in [n]$ so that $p = \langle \ldots i \ldots j \ldots k \ldots l \ldots \rangle$, $q = \langle \ldots j \ldots i \ldots l \ldots k \ldots \rangle$.*

4.4 A linear algorithm for checking the realizability of a pair of geometric permutations in $\mathbb{R}^2$

Let $\mathcal{P} = \{p, q\}$ be a pair of permutations on $[n]$. We want to check whether it is realizable in $\mathbb{R}^2$. It takes linear time to relabel the members of $[n]$ so that $p = (123\ldots n)$. Thus we shall assume that this is the case: $\mathcal{P} = \{p = (1, 2, \ldots, n), q = (q_1, q_2, \ldots, q_n)\}$. By Theorem 4.7, we need only to check whether q is decomposable. We present a linear algorithm that does this. Loosely speaking, we decompose q from left and from right, and then we either construct a decomposition of q, or prove that that q is not decopmposable.

Left decomposition. Define $K_L, M_L \subset [n]$ according to the following rules:

Step 1: $i = 1$; $K_L = M_L = \emptyset$.

Step 2: If $\max\{q_j | j \in K_L\} < q_i < \min\{q_j | j \in M_L\}$, or if K_L or M_L is $\emptyset$, mark i as "current". Otherwise STOP.

Step 3: If $i = 1$, $i := 2$ and go to step 2. If $i > 1$, continue.

Step 4: If $q_{i-1} < q_i$, add $i - 1$ to K_L. If $q_{i-1} > q_i$, add $i - 1$ to M_L.

Step 5: If $i = n$, STOP. If $i < n$, $i := i + 1$, and go to step 2.

Right decomposition. Define $K_R, M_R \subset [n]$ according to the following rules:

Step 1: $i = n$; $K_R = M_R = \emptyset$.

Step 2: If $\max\{q_j | j \in M_R\} < q_i < \min\{q_j | j \in K_R\}$, or if K_R or M_R is $\emptyset$, mark i as "current". Otherwise STOP.

Step 3: If $i = n$, $i := n - 1$ and go to step 2. If $i < n$, continue.

Step 4: If $q_i < q_{i+1}$, add $i + 1$ to K_R. If $q_i > q_{i+1}$, add $i - 1$ to M_R.

Step 5: If $i = 1$, STOP. If $i > 1$, $i := i - 1$, and go to step 2.

Denote by λ the "current" i on the termination of the left decomposition, and denote by ρ the "current" i on the termination of the right decomposition. Note that the subsequences $q|_{K_L}$ and $q|_{K_R}$ are ascending, and the subsequences $q|_{M_L}$ and $q|_{M_R}$ are descending. Moreover, λ is greater than all the members of $q|_{K_L}$ and less than all the members of $q|_{M_L}$; ρ is less than all the members of $q|_{K_R}$ and greater than all the members of $q|_{M_R}$. Note also that always $\lambda \geq 2$ and $\rho \leq n - 1$.

Example. Let $q = (928745361)$.

The left decomposition:	The right decomposition:
$\boxed{9}\ 2\ 8\ 7\ 4\ 5\ 3\ 6\ 1$	$9\ 2\ 8\ 7\ 4\ 5\ 3\ 6\ \boxed{1}$
$\overline{9}\ \boxed{2}\ 8\ 7\ 4\ 5\ 3\ 6\ 1$	$9\ 2\ 8\ 7\ 4\ 5\ 3\ \boxed{6}\ \underline{1}$
$\overline{9}\ \underline{2}\ \boxed{8}\ 7\ 4\ 5\ 3\ 6\ 1$	$9\ 2\ 8\ 7\ 4\ 5\ \boxed{3}\ \overline{6}\ \underline{1}$
$\overline{9}\ \underline{2}\ \overline{8}\ \boxed{7}\ 4\ 5\ 3\ 6\ 1$	$9\ 2\ 8\ 7\ 4\ \boxed{5}\ \underline{3}\ \overline{6}\ \underline{1}$
$\overline{9}\ \underline{2}\ \overline{8}\ \overline{7}\ \boxed{4}\ 5\ 3\ 6\ 1$	$9\ 2\ 8\ 7\ \boxed{4}\ \overline{5}\ \underline{3}\ \overline{6}\ \underline{1}$
$\overline{9}\ \underline{2}\ \overline{8}\ \overline{7}\ \underline{4}\ \boxed{5}\ 3\ 6\ 1$	

Here $K_L = \{2, 5\}$, $q|_{K_L} = (24)$; $M_L = \{1, 3, 4\}$, $q|_{M_L} = (987)$; $\lambda = 6$, $q_\lambda = 5$; and $K_R = \{6, 8\}$, $q|_{K_R} = (56)$; $M_R = \{7, 9\}$, $q|_{M_R} = (31)$; $\rho = 5$, $q_\rho = 4$, see Figure 4.9.

In order to establish whether q is decomposable we need to analyze the mutual position of λ and ρ (and, probably, of some neighbouring members). If $\lambda \neq \rho$, we shall assume that $q_\lambda < q_\rho$. There is no loss of generality since if $q_\lambda > q_\rho$ we can consider the permutation q' defined by $q'_i = n - q_i$: then $q'_\lambda < q'_\rho$, and it is clear that q is decomposable if and only if q' is decomposable. If $K_L \neq \emptyset$, denote $\lambda' = \max\{i : i \in K_L\}$; if $M_L \neq \emptyset$, denote $\lambda'' = \max\{i : i \in M_L\}$; if $K_R \neq \emptyset$, denote $\rho' = \min\{i : i \in K_R\}$; if $M_R \neq \emptyset$, denote $\rho'' = \min\{i : i \in M_R\}$.

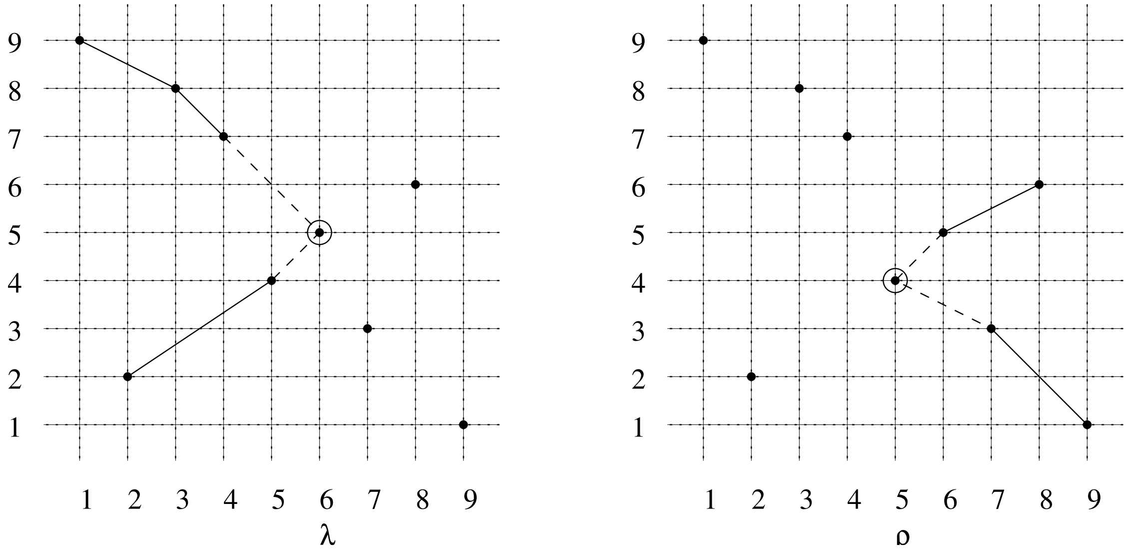

Figure 4.9: Left and right decompositions of $p = (928745361)$.

Claim 4.10 *Compare λ and ρ obtained in left and right decompositions. If $\lambda \neq \rho$, assume that $q_\lambda < q_\rho$. Then:*

1. *If $\lambda > \rho - 1$ then q is decomposable.*
2. *If $\lambda = \rho - 1$, then $M_L \neq \emptyset$ and $M_R \neq \emptyset$. If $q_{\lambda''} > q_{\rho''}$ then q is decomposable, and If $q_{\lambda''} < q_{\rho''}$ then q is not decomposable.*
3. *If $\lambda < \rho - 1$ then q is not decomposable.*

Proof

1. $\lambda > \rho - 1$ (or: $\lambda \geq \rho$).

   If $\lambda = \rho$, define $K = K_L \cup \{\lambda\} \cup K_R$, $M = M_L \cup M_R$ (of course, it is also possible to define λ to be a member of M instead of K).

   Suppose $\lambda > \rho$. Then, since $q_\lambda < q_\rho$, $\rho \in M_L$ and $\lambda \in M_R$. We assert that $i \in K_L \Rightarrow i < \rho$ and that $i \in K_R \Rightarrow i > \lambda$. We prove the first assertion: suppose $i \in K_L$ (this implies $i < \lambda$) and $i > \rho$. Then $i \in K_R$ or $i \in M_R$. However, both these possibilities contradict $q_i < q_\lambda < q_\rho$:

if $i \in K_R$, it must be $q_i > q_\rho$; if $i \in M_R$, it must be $q_i > q_\lambda$ (since $\lambda \in M_R$). The second assertion can be proved in a similar way. Define $K = K_L \cup K_R$, $M = M_L \cup M_R$. Then (K, M) is a decomposition of q.

2. $\lambda = \rho - 1$.

   If $M_L = \emptyset$, then it is possible to continue the left decomposition by defining ρ to be "current", in contradiction to the definition of λ. Therefore $M_L \neq \emptyset$. For a similar reason, $M_R \neq \emptyset$.

   If $q_{\lambda''} > q_{\rho''}$, just define $K = K_L \cup K_R$, $M = M_L \cup M_R$, and then (K, M) is a decomposition of q.

   If $q_{\lambda''} < q_{\rho''}$, we have $\lambda'' < \lambda < \rho < \rho''$ with $q_\lambda < q_{\lambda''} < q_{\rho''} < q_\rho$. Thus the restriction of q to $\{\lambda'', \lambda, \rho, \rho''\}$ is not decomposable (see Observation 4.5), and therefore q is not decomposable.

3. $\lambda < \rho - 1$. Denote $\lambda^+ = \lambda + 1$, $\rho^- = \rho - 1$. Then $\lambda^+ \leq \rho^-$ (the equality is possible).

   We shall prove that in this case there are $i, j, k, l \in [n]$ so that $i < j < k < l$ with $q_j < q_i < q_l < q_k$ or $q_j > q_i > q_l > q_k$. Then, by Observation 4.5, q has a non-decomposabe subsequence, and therefore q is non-decomposable.

   First we note the following: if $q_\lambda > q_{\lambda^+}$ then $K_L \neq \emptyset$ and $q_\lambda > q_{\lambda'} > q_{\lambda^+}$, since otherwise it is possible to continue the left decomposition defining $i = \lambda^+$ to be "current". Using similar arguments, we can see that if $q_\lambda < q_{\lambda^+}$ then $M_L \neq \emptyset$ and $q_\lambda < q_{\lambda''} < q_{\lambda^+}$; if $q_\rho > q_{\rho^-}$ then $M_R \neq \emptyset$ and $q_\rho > q_{\rho''} > q_{\rho^-}$; if $q_\rho < q_{\rho^-}$ then $K_R \neq \emptyset$ and $q_\rho < q_{\rho'} < q_{\rho^-}$.

   The case splits into several subcases.

   (3.1) $q_{\lambda^+} < q_\lambda$.
   We have $K_L \neq \emptyset$ and $q_{\lambda^+} < q_{\lambda'} < q_\lambda$.

   Suppose first $q_{\rho^-} > q_\rho$.
   Since $q_\lambda < q_\rho$, we obtain $q_{\lambda^+} < q_{\rho^-}$. Therefore $\lambda^+ < \rho^-$.
   Now we have $\lambda < \lambda^+ < \rho^- < \rho$ with $q_{\lambda^+} < q_\lambda < q_\rho < q_{\rho^-}$, and q is not decomposable (see Figure 4.10).

   Now suppose $q_{\rho^-} < q_\rho$. Then $M_R \neq \emptyset$ and $q_{\rho^-} < q_{\rho''} < q_\rho$.

If $q_{\lambda'} < q_{\rho''}$ then $\lambda' < \lambda^+ < \rho < \rho''$ with $q_{\lambda^+} < q_{\lambda'} < q_{\rho''} < q_\rho$, and q is not decomposable (see Figure 4.11).

If $q_{\rho''} < q_{\lambda'}$ then $\lambda' < \lambda < \rho^- < \rho''$ with $q_\lambda > q_{\lambda'} > q_{\rho''} > q_{\rho^-}$, and q is not decomposable (see Figure 4.12).

(3.2) $q_\lambda < q_{\lambda^+} < q_\rho$.

We have $M_L \neq \emptyset$ and $q_\lambda < q_{\lambda''} < q_{\lambda^+}$.

If $q_{\rho^-} > q_\rho$, replacing q with q'' defined by $q''_i = n - q_{n-i}$ gives us the case (3.1) that has already been checked.

Suppose $q_{\rho^-} < q_\rho$. Then $M_R \neq \emptyset$ and $q_{\rho^-} < q_{\rho''} < q_\rho$.

If $q_{\lambda''} < q_{\rho''}$ then $\lambda'' < \lambda < \rho < \rho''$ with $q_\lambda < q_{\lambda''} < q_{\rho''} < q_\rho$, and q is not decomposable (see Figure 4.13).

If $q_{\rho''} < q_{\lambda''}$ then $\lambda'' < \lambda^+ < \rho^- < \rho''$ with $q_{\lambda^+} > q_{\lambda''} > q_{\rho''} > q_{\rho^-}$, and q is decomposable (here $\lambda^+ < \rho^-$ since $q_{\lambda^+} \neq q_{\rho^-}$) (see Figure 4.14).

(3.3) $q_\rho < q_{\lambda^+}$.

We have $M_L \neq \emptyset$ and $q_\lambda < q_{\lambda''} < q_{\lambda^+}$.

If $q_{\rho^-} > q_\lambda$, replacing q with q'' defined by $q''_i = n - q_{n-i}$ gives us the case (3.1) that have been already checked.

Suppose $q_{\rho^-} < q_\lambda$. Then $M_R \neq \emptyset$ and $q_{\rho^-} < q_{\rho''} < q_\rho$.

If $q_{\lambda''} < q_\rho$ then $\lambda'' < \lambda < \lambda^+ < \rho$ with $q_\lambda < q_{\lambda''} < q_\rho < q_{\lambda^+}$, and q is not decomposable (see Figure 4.15).

If $q_\rho < q_{\lambda''}$ then $\lambda'' < \lambda^+ < \rho^- < \rho$ with $q_{\lambda^+} > q_{\lambda''} > q_\rho > q_{\rho^-}$, and q is not decomposable (here $\lambda^+ < \rho^-$ since $q_{\lambda^+} \neq q_{\rho^-}$) (see Figure 4.16). ■

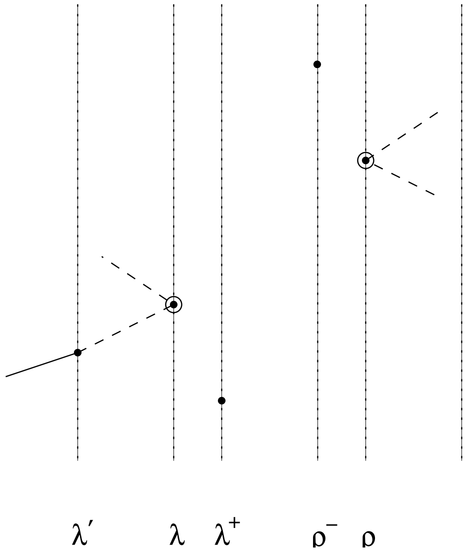

Figure 4.10: Case (3.1) with $q_{\rho^-} > q_\rho$.

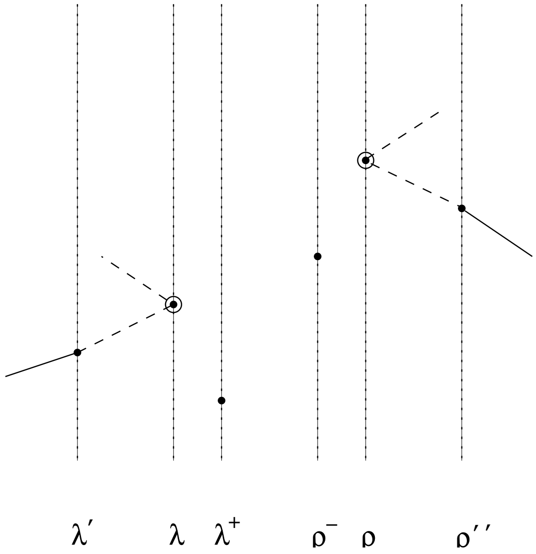

Figure 4.11: Case (3.1) with $q_{\rho^-} < q_\rho$ and $q_{\lambda'} < q_{\rho''}$.

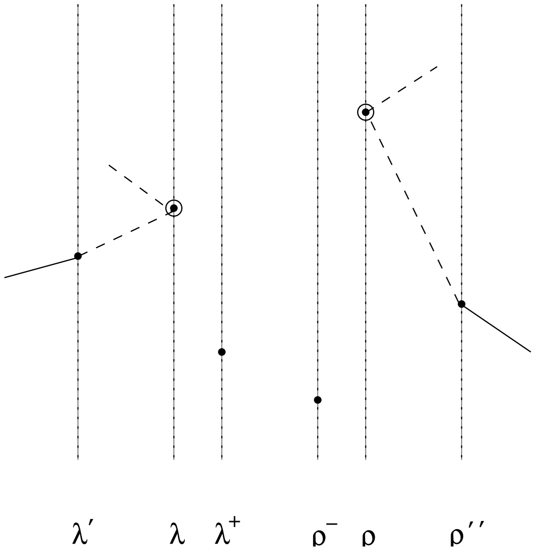

Figure 4.12: Case (3.1) with $q_{\rho^-} < q_\rho$ and $q_{\rho''} < q_{\lambda'}$.

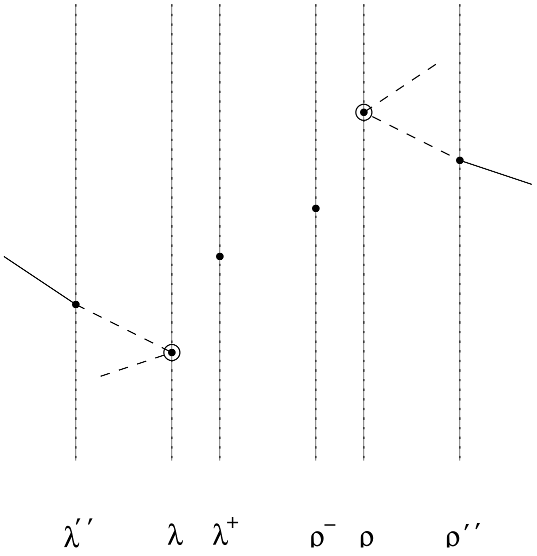

Figure 4.13: Case (3.2) with $q_{\rho^-} < q_\rho$ and $q_{\lambda''} < q_{\rho''}$

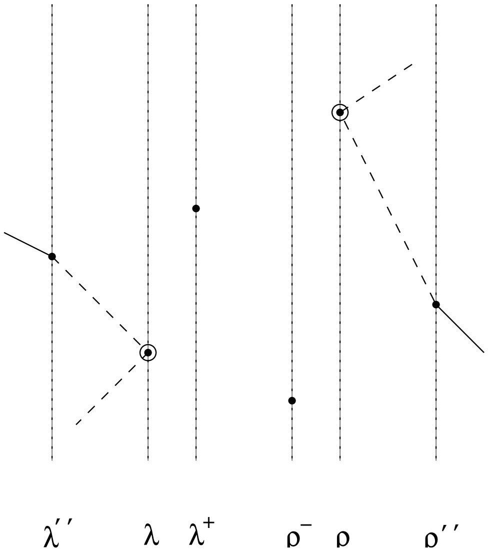

Figure 4.14: Case (3.2) with $q_{\rho^-} < q_\rho$ and $q_{\rho''} < q_{\lambda''}$

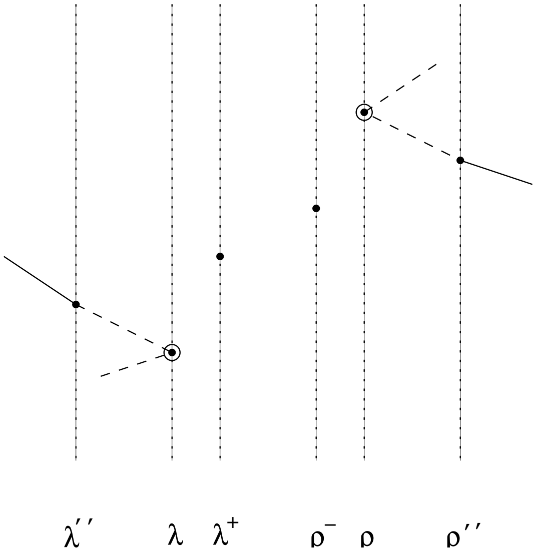

Figure 4.15: Case (3.3) with $q_{\rho^-} < q_\lambda$ and $q_{\lambda''} < q_\rho$

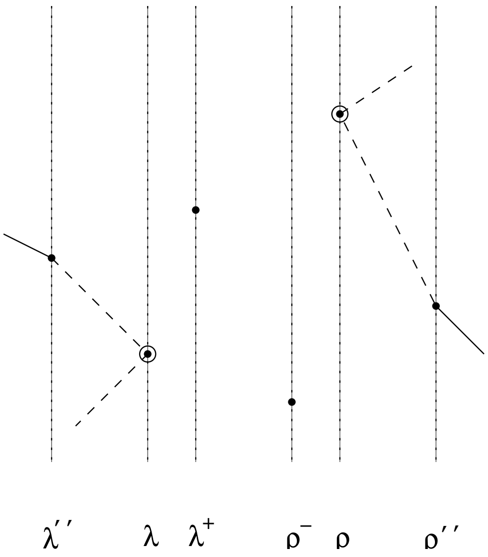

Figure 4.16: Case (3.3) with $q_{\rho^-} < q_\lambda$ and $q_\rho < q_{\lambda''}$

4.5 Two necessary conditions for realizability of families of geometric permutations in $\mathbb{R}^2$

Suppose $\mathcal{P} = \{p^1, p^2, \ldots, p^k\}$ is a family of permutations of $[n]$, and we want to know whether it is realizable in $\mathbb{R}^2$. For $k = 2$ the problem was solved in Section 4.3 (Theorems 4.7 and 4.9). We shall see that for $k > 2$ the situation is much more complicated. We describe some necessary conditions for realizability, and show that they are not sufficient.

Necessary condition 1

The following statement is clear:

Observation 4.11 (Necessary condition 1) *If $\mathcal{P}$ is realizable in $\mathbb{R}^2$, then no two members of $\mathcal{P}$ form a forbidden pair.*

The following example (taken from [1]) shows that this condition is not sufficient. Let $n \in \mathbb{N}$. For each $A \subseteq \{2, 3 \ldots, \lfloor n/2 \rfloor\}$, define the permutation $p = p_A$ on $[n]$ as follows:

If $j \in A$ or $n - j \in A$, then $p_j = n - j$; otherwise $p_j = j$.

For example, for $n = 6$, we have:

$p_{\emptyset} = (123456)$, $p_{\{2\}} = (153426)$, $p_{\{3\}} = (124356)$, $p_{\{2,3\}} = (154326)$.

In this way we obtain $2^{\lfloor n/2 \rfloor - 1}$ permutations. Since all of them begin with 1, no two of them are reverses of each other. Each two of them form a realizable pair: for p_A and p_B, where $A, B \subseteq \{2, 3, \ldots, \lfloor n/2 \rfloor\}$, let $K = \{i | i \in A \triangle B \text{ or } n - i \in A \triangle B\}$, $M = [n] \setminus K$. Then $p_A|_K = p_B|_K$ and $p_A|_M = -p_B|_M$, and thus the pair $\{p_A, p_B\}$ is realizable by Corollary 4.8.

Thus we have a family of $2^{\lfloor n/2 \rfloor - 1}$ pairwise realizable permutations. However, the entire family is forbidden since the maximal size of a realizable family in $\mathbb{R}^2$ is $2n - 2$ [9].

Necessary condition 2

Definition. Let $\mathcal{P} = (p^{i_1}, p^{i_2}, \ldots, p^{i_k})$ be a family of (directed) permutations on $[n]$, ordered by $p^{i_j} \ll p^{i_{j'}} \Leftrightarrow j < j'$. We say that the ordering $\ll$ is *good* if for each $a, b \in [n]$ with $p^{i_1} : (a \prec b)$ there is $y = y(a, b) \in [k]$ so that $p^{i_x} : (a \prec b)$ for $x = 1, 2, \ldots, y$, and $p^{i_x} : (b \prec a)$ for $x = y + 1, y + 2, \ldots, k$.

This can be reformulated in the following way: there are no $x, y, z \in [k]$, $x < y < z$, so that for some $a, b \in [n]$, $p^{i_x} : (a \prec b)$, $p^{i_y} : (b \prec a)$, $p^{i_z} : (a \prec b)$.

Claim 4.12 (Necessary condition 2) *Let $\mathcal{P} = \{\tilde{p}^1, \tilde{p}^2, \ldots, \tilde{p}^k\}$ be a family of undirected permutations ($\tilde{p}^j = \{p^j, -p^j\}$), realizable in $\mathbb{R}^2$. Then:*

1. *It is possible to choose a representative of each member of $\mathcal{P}$, and to order them using a good order $\ll$ (that is, the ordering of the family ($p^{i_1} \ll p^{i_2} \ll \cdots \ll p^{i_k}$) is a good one).*

2. *There are precisely $4k$ such good orderings.*

Proof.

1. Consider a realization of $\mathcal{P}$. Choose a direction on each transversal so that all the directions will be contained in an open semicircle Σ on the circle of directions. Order the transversals (with their permutations) according to the order of the appearance (for example, clockwise) of their directions on circle of directions.

   Let $a, b \in [n]$. Consider a line m that separates S_a from S_b, and translate it to the circle of directions. This line divides Σ into two sectors (see Figure 4.17). Then $a \prec b$ on the transversals that belong to one of them, and $b \prec a$ on the transversals that belong to another.

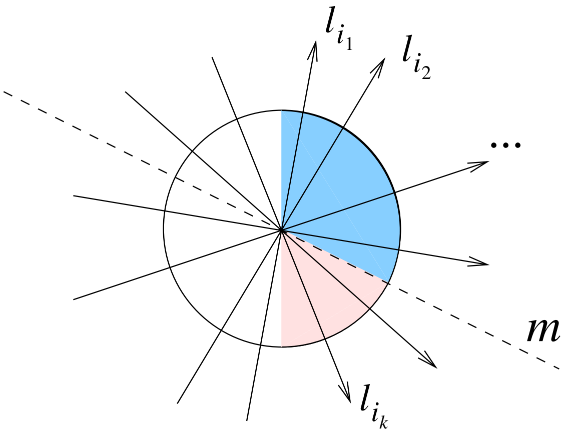

Figure 4.17: Illustration to the proof of Necessary condition 2 (Claim 4.12).

2. Suppose without loss of generality that ($p^1, p^2, \ldots, p^k$) is a good ordering of representatives of the members of $\mathcal{P}$. Then for each $j \in [k]$, the following orderings are also good:

- $(p^j,\, p^{j+1},\, \ldots,\, p^k,\, -p^1,\, -p^2,\, \ldots,\, -p^{j-1})$
- $(-p^j,\, -p^{j+1},\, \ldots,\, -p^k,\, p^1,\, p^2,\, \ldots,\, p^{j-1})$
- $(p^j,\, p^{j-1},\, \ldots,\, p^1,\, -p^k,\, -p^{k-1},\, \ldots,\, -p^{j+1})$
- $(-p^j,\, -p^{j-1},\, \ldots,\, -p^1,\, p^k,\, p^{k-1},\, \ldots,\, p^{j+1})$

We shall prove only that the first of these orderings, $(p^j,\, p^{j+1},\, \ldots,\, p^k,\, -p^1,\, -p^2,\, \ldots,\, -p^{j-1})$, is good (the remaining three orderings are proved in a similar way). If not, we have one of the following cases:

(a) There exist $a, b \in [n]$ and $x, y, z \in [k]$ and so that $j \leq x < y < z \leq k$ and $p^x : (a \prec b)$, $p^y : (b \prec a)$, $p^z : (a \prec b)$. This is impossible since $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering.

(b) There exist $a, b \in [n]$ and $x, y, z \in [k]$ and so that $j \leq x < y \leq k$, $1 \leq z \leq j-1$ and $p^x : (a \prec b)$, $p^y : (b \prec a)$, $-p^z : (a \prec b)$. This is impossible since then $p^z : (b \prec a)$, $p^x : (a \prec b)$, $p^y : (b \prec a)$ with $z < x < y$, and $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering.

(c) There exist $a, b \in [n]$ and $x, y, z \in [k]$ and so that $j \leq x \leq k$, $1 \leq y < z \leq j-1$ and $p^x : (a \prec b)$, $-p^y : (b \prec a)$, $-p^z : (a \prec b)$. This is impossible since then $p^y : (a \prec b)$, $p^z : (b \prec a)$, $p^x : (a \prec b)$ with $y < z < x$, and $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering.

(d) There exist $a, b \in [n]$ and $x, y, z \in [k]$ and so that $1 \leq x < y < z \leq j-1$ and $-p^x : (a \prec b)$, $-p^y : (b \prec a)$, $-p^z : (a \prec b)$. This is impossible since then $p^x : (b \prec a)$, $p^y : (a \prec b)$, $p^z : (b \prec a)$ with $x < y < z$, and $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering.

Thus we have at least $4k$ good orderings of $\mathcal{P}$. It remains to prove that *only* these orderings of are good.

Assume again that $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering of $\mathcal{P}$, and let $x, y, z \in [k]$, $x < y < z$.

Since p^x and p^z are representatives of distinct undirected permutations and thus are not reverses of each other, there are $a, b \in [n]$ so that $p^x : (a \prec b)$ and $p^z : (a \prec b)$. Since $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering, and $x < y < z$, we have also $p^y : (a \prec b)$.

Since p^x and p^y are representatives of distinct undirected permutations, there are $c, d \in [n]$ so that $p^x : (c \prec d)$ and $p^y : (d \prec c)$. Since $(p^1,\, p^2,\, \ldots,\, p^k)$ is a good ordering, and $x < y < z$, we have $p^z : (d \prec c)$.

Since p^y and p^z are representatives of distinct undirected permutations, there are $e, f \in [n]$ so that $p^y : (e \prec f)$ and $p^z : (f \prec e)$. Since $(p^1, p^2, \ldots, p^k)$ is a good ordering, and $x < y < z$, we have $p^x : (e \prec f)$.

To summarize: there exist $a, b, c, d, e, f \in [n]$ (not necessarily all of them distinct) such that:

$p^x : (a \prec b)$, $p^y : (a \prec b)$, $p^z : (a \prec b)$;

$p^x : (c \prec d)$, $p^y : (d \prec c)$, $p^z : (d \prec c)$;

$p^x : (e \prec f)$, $p^y : (e \prec f)$, $p^z : (f \prec e)$.

We claim that in any good ordering $\ll$ of $\mathcal{P}$, the representatives of $\tilde{p}^x$, $\tilde{p}^y$, $\tilde{p}^z$ appear in one of the following 12 ways:

$$\begin{array}{ll}
p^x \ll p^y \ll p^z, & p^x \ll -p^z \ll -p^y, \\
-p^x \ll -p^y \ll -p^z, & -p^x \ll p^z \ll p^y, \\
p^y \ll p^z \ll -p^x, & p^y \ll p^x \ll -p^z, \\
-p^y \ll -p^z \ll p^x, & -p^y \ll -p^x \ll p^z, \\
p^z \ll -p^x \ll -p^y, & p^z \ll p^y \ll p^x, \\
-p^z \ll p^x \ll p^y, & -p^z \ll -p^y \ll -p^x.
\end{array}$$

All the other cases are impossible. For example:

$p^x \ll p^z \ll p^y$ is impossible since $p^x : (e \prec f)$, $p^z : (f \prec e)$, $p^y : (e \prec f)$.

$p^x \ll p^z \ll -p^y$ is impossible since $p^x : (c \prec d)$, $p^z : (d \prec c)$, $-p^y : (c \prec d)$.

$p^x \ll -p^y \ll p^z$ is impossible since $p^x : (a \prec b)$, $-p^y : (b \prec a)$, $p^z : (a \prec b)$.

Now we shall prove our claim by induction on k. We have already proved it for $k = 3$ (see the 12 possibilities above). Suppose that we have proved that for some $k \geq 3$ the only good orderings are:

(a) $(p^1, p^2, \ldots, p^k)$
(b) $(-p^1, -p^2, \ldots, -p^k)$
(c) $(p^k, p^{k-1}, \ldots, p^1)$
(d) $(-p^k, -p^{k-1}, \ldots, -p^1)$
(e) $(p^j, p^{j+1}, \ldots, p^k, -p^1, -p^2, \ldots, -p^{j-1})$ with $j > 1$
(f) $(-p^j, -p^{j+1}, \ldots, -p^k, p^1, p^2, \ldots, p^{j-1})$ with $j > 1$

(g) $(p^j, p^{j-1}, \dots, p^1, -p^k, -p^{k-1}, \dots, -p^{j+1})$ with $j < k$

(h) $(-p^j, -p^{j-1}, \dots, -p^1, p^k, p^{k-1}, \dots, p^{j+1})$ with $j < k$

These are precisely the four types of orderings listed above, but this time we consider separately the orderings in which the first and the last permutations are $\pm p^1$, $\pm p^k$. There is one such ordering of each of the types (a)–(d), and $k-1$ orderings of each of the types (e)–(h).

Now suppose we have a good ordering of representatives of $\{\tilde{p}^1, \tilde{p}^2, \dots, \tilde{p}^{k+1}\}$. Any restriction of a good ordering is clearly also good, thus any good ordering for $k+1$ permutations can be obtained by adding of $\pm p^{k+1}$ to one of the $4k$ orderings of representatives of $\{\tilde{p}^1, \tilde{p}^2, \dots, \tilde{p}^k\}$ among the listed above. We claim that for (a)–(d) this can be done in two ways, and for (e)–(h), in one way:

(a) $(p^1, p^2, \dots, p^k) \longrightarrow$
$\longrightarrow (p^1, p^2, \dots, p^k, p^{k+1})$ or $(-p^{k+1}, p^1, p^2, \dots, p^k)$

(b) $(-p^1, -p^2, \dots, -p^k) \longrightarrow$
$\longrightarrow (-p^1, -p^2, \dots, -p^k, -p^{k+1})$ or $(p^{k+1}, -p^1, -p^2, \dots, -p^k)$

(c) $(p^k, p^{k-1}, \dots, p^1) \longrightarrow$
$\longrightarrow (p^{k+1}, p^k, p^{k-1}, \dots, p^1)$ or $(p^k, p^{k-1}, \dots, p^1, -p^{k+1})$

(d) $(-p^k, -p^{k-1}, \dots, -p^1) \longrightarrow$
$\longrightarrow (-p^{k+1}, -p^k, -p^{k-1}, \dots, -p^1)$ or $(-p^k, -p^{k-1}, \dots, -p^1, p^{k+1})$

(e) $(p^j, p^{j+1}, \dots, p^k, -p^1, -p^2, \dots, -p^{j-1})$ with $j > 1 \longrightarrow$
$\longrightarrow (p^j, p^{j+1}, \dots, p^k, p^{k+1}, -p^1, -p^2, \dots, -p^{j-1})$

(f) $(-p^j, -p^{j+1}, \dots, -p^k, p^1, p^2, \dots, p^{j-1})$ with $j > 1 \longrightarrow$
$\longrightarrow (-p^j, -p^{j+1}, \dots, -p^k, -p^{k+1}, p^1, p^2, \dots, p^{j-1})$

(g) $(p^j, p^{j-1}, \dots, p^1, -p^k, -p^{k-1}, \dots, -p^{j+1})$ with $j < k \longrightarrow$
$\longrightarrow (p^j, p^{j-1}, \dots, p^1, -p^{k+1}, -p^k, -p^{k-1}, \dots, -p^{j+1})$

(h) $(-p^j, -p^{j-1}, \dots, -p^1, p^k, p^{k-1}, \dots, p^{j+1})$ with $j < k \longrightarrow$
$\longrightarrow (-p^j, -p^{j-1}, \dots, -p^1, p^{k+1}, p^k, p^{k-1}, \dots, p^{j+1})$

This gives $2 \cdot 4 + 1 \cdot (k-1) = 4 \cdot (k+1)$ orderings. We shall prove that these are the only ways to add $\pm p^{k+1}$ for cases (a) and (e) (for other cases this can be proved similarly).

(a) By the assumption, $(p^1, p^2, \dots, p^k)$ is a good ordering. We need to add p^{k+1} or $-p^{k+1}$ to it. According to what we have proved for 3 permutations (see 12 cases on page 49) with $x = 1$, $y = k$, $z = k + 1$, the only possibilities for p^1, p^k, $\pm p^{k+1}$ with $p^1 \ll p^k$ are $p^1 \ll p^k \ll p^{k+1}$ and $-p^{k+1} \ll p^1 \ll p^k$. This yields $(p^1, p^2, \dots, p^k, p^{k+1})$ and $(-p^{k+1}, p^1, p^2, \dots, p^k)$.

(e) By the assumption, $(p^j, p^{j+1}, \dots, p^k, -p^1, -p^2, \dots, -p^{j-1})$ (with $j > 1$) is a good ordering. We need to add p^{k+1} or $-p^{k+1}$ to it. According to what we have proved for 3 permutations with $x = 1$, $y = k$, $z = k + 1$, the only possibility for $-p^1$, p^k, $\pm p^{k+1}$ with $p^k \ll -p^1$ is $p^k \ll p^{k+1} \ll -p^1$.

This yields $(p^j, p^{j+1}, \dots, p^k, p^{k+1}, -p^1, -p^2, \dots, -p^{j-1})$. ■

We saw in the proof that for a realizable family of permutations, a good ordering of their representatives corresponds to a natural ordering of their directions on the circle of directions. In this case, $4k$ good orderings correspond to the choice of the first permutation, its orientation, and clockwise or counterclockwise ordering at the circle of directions, and they can be obtained from each other by rotation and/or reflection of the plane.

Necessary Condition 2 is also not sufficient. Consider the following family of permutations: begin with $p^1 = (123\dots n)$, then move 1 by one to the right until it reaches the last position, then move 2 by one to the right until it reaches next to the last position, and so on (each i reaches the position $n-i+1$). Each time just one pair of sets interchange, and no pair interchanges twice (this means that this family satisfies Necessary Condition 2). The last permutation will be $(n-1, n, n-2, n-3, \dots, 2, 1)$ (only $n-1$ and n remain not interchanged). For example, for $n = 4$ we have the following sequence of permutations:

$p^1 = (1234)$, $p^2 = (2134)$, $p^3 = (2314)$, $p^4 = (2341)$, $p^5 = (3241)$, $p^6 = (3421)$.

There are $\binom{n}{2}$ permutations in this family. Again, by the result from [9], this means that such a family is forbidden (for $n \geq 5$).

Another way to see that this family is forbidden (for $n \geq 4$) is to note that $p^2 = (2, 1, 3, 4, \dots, n-1, n)$ and $p^{\binom{n}{2}} = (n-1, n, n-2, n-3, \dots, 2, 1)$, and these two permutations form a forbidden pair: note the restriction to $\{1, 2, n-1, n\}$. Thus we see that Condition 1 does not follow from Condition 2.

In fact, Conditions 1 and 2 are independent. Consider a family $\mathcal{P} = \{\tilde{p}^1, \tilde{p}^2, \tilde{p}^3\} = \{\langle 1234\rangle, \langle 1342\rangle, \langle 1423\rangle\}$ of non-directed permutations. This family satisfies Condition 1 (its members are pairwise compatible), but does not satisfy Condition 2: if we try to find a good ordering $p^1 \ll p^2 \ll p^3$ of representatives, it is possible to assume that $p^1 = (1234)$, it is also possible to assume that $p^2 = \pm(1342)$, therefore $p^3 = \pm(1423)$. There are four ways to choose representatives of $\tilde{p}^2$ and $\tilde{p}^3$ and to obtain thus an ordering of this family of non-directed permutations. None of them is good as for each of them there are two sets that violate Condition 2:
$(1\boxed{2}\boxed{3}4), (1\boxed{3}4\boxed{2}), (14\boxed{2}\boxed{3})$;
$(1\boxed{2}3\boxed{4}), (13\boxed{4}\boxed{2}), (3\boxed{2}\boxed{4}1)$;
$(\boxed{1}\boxed{2}34), (\boxed{2}43\boxed{1}), (\boxed{1}4\boxed{2}3)$;
$(12\boxed{3}\boxed{4}), (2\boxed{4}\boxed{3}1), (\boxed{3}2\boxed{4}1)$.

4.6 The necessary conditions are not sufficient

We show that Necessary Conditions 1 and 2 from Section 4.5, even together, are not sufficient. Consider the following family of three permutations for six sets: $\mathcal{P} = \{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$. This family satisfies both the necessary conditions: it is easy to check that each two of them form a realizable pair; and $(p^1 = (123456), p^2 = (412563), p^3 = (541632))$ is a good ordering of their representatives. However, we shall prove that $\mathcal{P}$ is forbidden in $\mathbb{R}^2$.

Claim 4.13 *The family of permutations* $\mathcal{P} = \{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$ *is not realizable in* $\mathbb{R}^2$.

We shall use the following simple fact.

Observation 4.14 *Let* $p^1 = (123)$, $p^2 = (132)$ *be two directed geometric permutations of* $\{A_1, A_2, A_3\}$, *realized by directed transversal lines* l_1 *and* l_2. *Then* A_1 *crosses quadrant* III.[1]

Proof of Observation 4.14 We need to show that $A_1 \cap l_1 \subset l_1^-$ and that $A_1 \cap l_2 \subset l_2^-$.

---

[1] Quadrants have been defined on page 28.

Suppose $A_1 \cap l_1^+ \neq \emptyset$. Then $A_2 \cap l_1 \subset l_1^+$ and $A_3 \cap l_1 \subset l_1^+$.

It is clear that $O \notin A_3$. If $A_3 \cap l_2 \subset l_2^+$ then also $A_2 \cap l_2 \subset l_2^+$, and $A_3 \cap A_2 \neq \emptyset$. If $A_3 \cap l_2 \subset l_2^-$ then also $A_1 \cap l_2 \subset l_2^-$, and $A_3 \cap A_1 \neq \emptyset$ (see Figure 4.18).

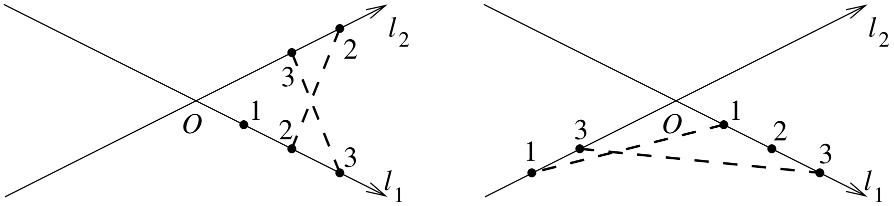

Figure 4.18: Illustration of the proof of Observation 4.14.

Thus $A_1 \cap l_1 \subset l_1^-$. By symmetry, also $A_1 \cap l_2 \subset l_2^-$. ■

Proof of Claim 4.13 Suppose $(\mathcal{F} = \{A_1, A_2, A_3, A_4, A_5, A_6\}, \mathcal{L} = \{l_1, l_2, l_3\})$ is a realization of $\mathcal{P}$ (l_j is a directed transversal that induces p^j). Since (p^1, p^2, p^3) is a good ordering, we can assume without loss of generality (see remarks after the proof of Claim 4.12) that (l_1, l_2, l_3) is the order in which these lines appear on the circle of directions, see Figure 4.19.

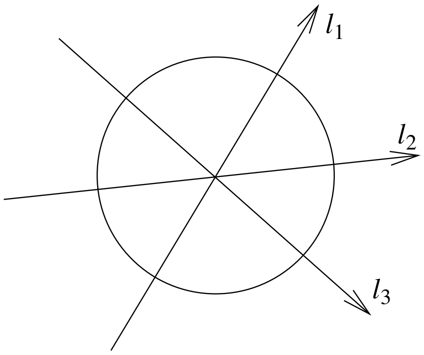

Figure 4.19: The order of the directions of the lines l_1, l_2, l_3.

For $i, j \in \{1, 2, 3\}$ denote $O_{ij} = O_{ji} = l_i \cap l_j$. It is possible to assume that O_{12}, O_{13}, O_{23} are distinct points. There are two possible cases: the first is $l_1 : (O_{12} \prec O_{13})$, $l_2 : (O_{21} \prec O_{23})$, $l_3 : (O_{31} \prec O_{32})$; the second is $l_1 : (O_{13} \prec O_{12})$, $l_2 : (O_{23} \prec O_{21})$, $l_3 : (O_{32} \prec O_{31})$, see Figure 4.20.

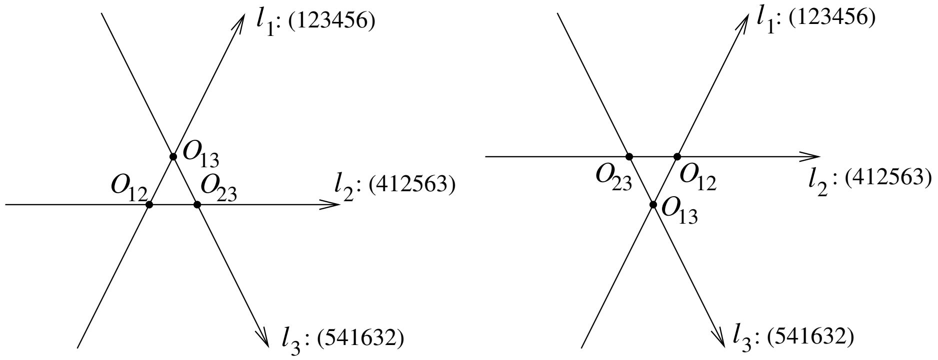

Figure 4.20: Two cases for the lines l_1, l_2, l_3.

However, the second case can be obtained from the first by relabeling the sets $1 \mapsto 4$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 5$, $5 \mapsto 6$, $6 \mapsto 3$, the transversals $l_1 \mapsto l_2$, $l_2 \mapsto l_3$, $l_3 \mapsto -l_1$, and the intersection points $O_{12} \mapsto O_{23}$, $O_{13} \mapsto O_{12}$, $O_{23} \mapsto O_{13}$. Therefore we shall consider only the first case.

We claim that
$l_1 : 1 \prec 2 \prec 3 \prec O_{12} \prec O_{13} \prec 4 \prec 5 \prec 6$,
$l_2 : 4 \prec 1 \prec 2 \prec O_{21} \prec O_{23} \prec 5 \prec 6 \prec 3$,
$l_3 : 5 \prec 4 \prec 1 \prec O_{31} \prec O_{32} \prec 6 \prec 3 \prec 2$ (see Figure 4.21).

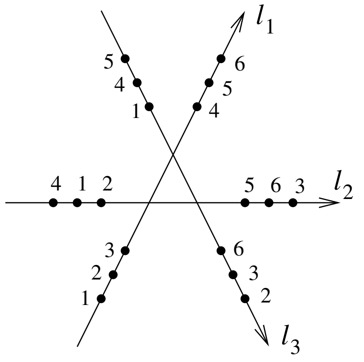

Figure 4.21: Position of the sets A_i at the lines l_j, about the points of intersection of the transversals.

In order to prove this, we use Observation 4.14 several times:

Since $l_1 : 3 \prec 4 \prec 5$ and $-l_2 : 3 \prec 5 \prec 4$, we have by Observation 4.14 $l_1 : 3 \prec O_{12}$ (and $l_2 : O_{21} \prec 3$). Therefore $l_1 : 1 \prec 2 \prec 3 \prec O_{12}$. Similarly we obtain $l_2 : O_{23} \prec 5 \prec 6 \prec 3$ and $l_3 : 5 \prec 4 \prec 1 \prec O_{31}$.

Since $-l_1 : 4 \prec 2 \prec 1$ and $l_3 : 4 \prec 1 \prec 2$, we have by Observation 4.14 $l_1 : O_{13} \prec 4$ (and $l_3 : 4 \prec O_{31}$). Therefore $l_1 : O_{13} \prec 4 \prec 5 \prec 6$. Similarly we obtain $l_2 : 4 \prec 1 \prec 2 \prec O_{21}$ and $l_3 : O_{32} \prec 6 \prec 3 \prec 2$.

Using an affine transformation, we can assume that the angle between each two transversals is 60°. Then the realization must look similar to the situation on Figure 4.22, with arcs replaced by convex sets. It is also possible to assume that the sets are segments with endpoints on transversal lines: let $P \in A_1 \cap l_1$, $Q \in A_1 \cap l_3$, and let R be the point of intersection of the segment PQ with l_2. Then $l_2 : (R \prec O_{12})$ and therefore the set A_1 can be replaced by the segment PQ. Applying the same argument to all the sets in $\mathcal{F}$ we conclude that it can be assumed that all the members of $\mathcal{F}$ are segments.

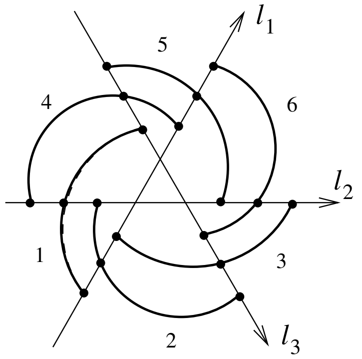

Figure 4.22: A "realization" of $\{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$ by arcs.

However, there is no realization of $\mathcal{F}$ with segments with endpoints on transversal lines. Assume that there is such a realization, and denote by δ the minimal distance between an endpoint of one of them and one of the points of the intersection of the transversals. Without loss of generality, $\delta = \text{dist}(A_1 \cap l_3, O_{13})$ or $\delta = \text{dist}(A_2 \cap l_2, O_{21})$ (other possibilities can be obtained from one of these by relabeling the sets and the transversals). Then (since all the angles between the transversals are 60°) in the first case $\text{dist}(A_2 \cap l_2, O_{21}) > \delta$ and A_1 misses l_1; in the second case $\text{dist}(A_3 \cap l_1, O_{12}) > \delta$ and A_2 misses l_3 (see Figure 4.23). It follows that the family $\mathcal{P}$ is not realizable in $\mathbb{R}^2$. ■

Remark. A similar construction was used by Sharir and Smorodinsky in their study of neighboring sets in geometric permutations [23]. Though

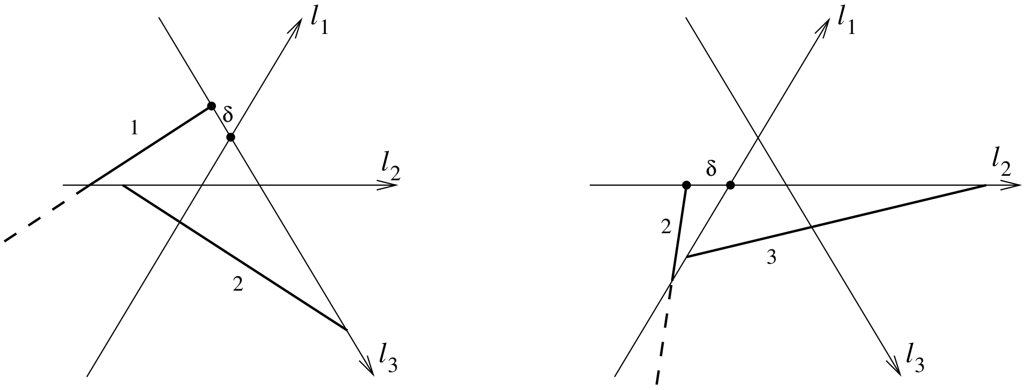

Figure 4.23: The proof of the impossibility of realization the family $\{\langle 123456\rangle, \langle 412563\rangle, \langle 541632\rangle\}$ in $\mathbb{R}^2$.

used in slightly different contexts, the two constructions point to constraints on geometric permutations for one family of disjoint convex sets.

4.7 Concluding remarks

The most obvious open problem related to the results from this chapter is the following: given a family of permutations $\mathcal{P}$, how to check whether it is realizable in $\mathbb{R}^2$? Theorems 4.7 and 4.9 solve this problem for families of two permutations, but the situation seems to be more complicated for larger families of permutations.

The Helly-type condition in Theorem 4.9 suggests a question: does a similar condition exists for larger families of permutations? Is it true that for each k (the number of permutations) there is [a Helly number] $\alpha = \alpha(k)$ so that if any restriction of the members of $\mathcal{P}$ to α sets is realizable then the whole family $\mathcal{P}$ is realizable? As a first step it is possible to try to solve this problem with fixed transversal lines.

Bibliography

[1] H. L. Abbott and M. Katchalski. On permutations with certain constraints. *Ars Combinatorica* **6** (1978), 19-26.

[2] B. Aronov and S. Smorodinsky. On geometric permutations induced by lines transversal through a fixed point. *Proc. 16th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) (2005).*

[3] A. Asinowski and M. Katchalski. Forbidden families of geometric permutations in $\mathbb{R}^d$. *Discrete Comput. Geom.* **34** (2005), 1-10.

[4] A. Asinowski and M. Katchalski. The maximal number of geometric permutations for n disjoint translates of a convex set in $\mathbb{R}^3$ is $\Omega(n)$. Accepted for publication in *Discrete Comput. Geom.*

[5] O. Cheong, X. Goaoc, and A. Holmsen. Hadwiger and Helly-type theorems for disjoint unit spheres in $\mathbb{R}^3$. In *Proc. of ACM Symposium on Computational Geometry*, 2005; invited to the special issue of *Computational Geometry: Theory and Applications* on SoCG, 2005.

[6] O. Cheong, X. Goaoc, and H.-S. Na. Geometric permutations of disjoint unit spheres, *Comput. Geom.: Theory and Applications* **30** (2005), 253-270.

[7] L. Danzer, B. Grünbaum, and V. Klee. Helly's theorem and its relatives. *Proc. Sympos. Pure Math.* **7** (Convexity) Amer. Math, Soc., Providence, RI, (1962), 101-180.

[8] J. Eckhoff. Helly, Radon and Carathéodory type theorems. In *Handbook of Convex Geometry*, North-Holland, 1993, pp. 389-448.

[9] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex nonintersecting sets in the plane is $2n-2$. *Discrete Comput. Geom.* **5** (1990), 35-42.

[10] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In *New Trends in Discrete and Computational Geometry*, J. Pach, ed., vol. 10 of Algorithms and Combinatorics, Springer-Verlag, Heidelberg, Germany, 1993, pp. 163-198.

[11] B. Grünbaum. On common transversals. *Arch. Math.* **9** (1958), 465-469.

[12] H. Hadwiger, H. Debrunner, and V. Klee. *Combinatorial Geometry in the Plane.* Holt, Rinehart and Winston, New York, 1964.

[13] A. Holmsen, M. Katchalski, T. Lewis. A Helly-type theorem for line transversals to disjoint unit balls. *Discrete Comput. Geom.* **29** (2003), 595-602.

[14] A. Holmsen, J. Matoušek. No Helly theorem for stabbing translates by lines in $\mathbb{R}^3$. *Discrete Comput. Geom.* **31** (2004), 405-410.

[15] M. Katchalski. A conjecture of Grünbaum on common transversals. *Math. Scand.* **59** (1986), 192-198.

[16] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations and common transversals. *Discrete Comput. Geom.* **1** (1986), 371-377.

[17] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations of disjoint translates of convex sets. *Discrete Math.* **65** (1987), 249-260.

[18] M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint sets. *Discrete Comput. Geom.* **7** (1992), 197-206.

[19] M. Katchalski, T. Lewis, and J. Zaks. Geometric permutations for convex sets, *Discrete Math.* **54** (1985), 271-284.

[20] M. Katchalski, S. Suri, and Y. Zhou. A constant bound for geometric permutations of disjoint unit balls, *Discrete Comput. Geom.* **29** (2003), 161-173.

[21] M. J. Katz and K. R. Varadarajan. A tight bound on the number of geometric permutations of convex fat objects in $\mathbb{R}^d$, *Discrete Comput. Geom.* **26** (2001), 543-548.

[22] J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. Ann.* **83** (1921), 113-115.

[23] M. Sharir and S. Smorodinsky. Neighbors in geometric permutations. *Discrete Math.* **268** (2003), 327-335.

[24] Z. E. Stankova. Forbidden subsequences. *Discrete Math.* **132** (1994), 291-316.

[25] S. Smorodinsky, J. S. B. Mitchell, and M. Sharir. Sharp bounds on geometric permutations of pairwise disjoint dalls in $\mathbb{R}^d$. *Discrete Comput. Geom.* **23** (2000), 247-259.

[26] H. Tverberg. Proof of Grünbaum's conjecture on common transversals for translates. *Discrete Comput. Geom.* **4**, (1989), 191-203.

[27] R. Wenger. Helly-type theorems and geometric transversals. In *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, Boca Raton, 2004, pp. 73-96.

[28] R. Wenger. Upper bounds on geometric permutations for Convex Sets. *Discrete Comput. Geom.* **5**, (1990), 27-33.

[29] R. Wenger. Progress in geometric transversal theory. In *Advances in Discrete and Computational Geometry*, Contemporary Mathematics, Vol 223, Amer. Math. Soc., Providence, RI, (1999), pp. 375-393.