

The maximal number of geometric permutations
for n disjoint translates of a convex set in \mathbb{R}^3
is $\Omega(n)$

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Abstract

A geometric permutation induced by a transversal line of a finite family of disjoint convex sets in \mathbb{R}^d is the order in which the transversal meets the members of the family. It is known that the maximal number of geometric permutations in families of n disjoint translates of a convex set in \mathbb{R}^2 is 3. We prove that for $d \geq 3$ the maximal number of geometric permutations for such families in \mathbb{R}^d is $\Omega(n)$.

1 Introduction

Let $\mathcal{F} = \{A_1, A_2, \dots, A_n\}$ be a finite family of n pairwise disjoint convex sets in \mathbb{R}^d . A line l is a *transversal* of \mathcal{F} if it intersects all the members of \mathcal{F} . Each non-directed transversal intersects the members of \mathcal{F} in an order which can be described by a pair of permutations of $\{1, 2, \dots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation*.

There are several results concerning the maximal number of geometric permutations for families of n disjoint convex sets in \mathbb{R}^d (we shall denote it by $g_d(n)$). Among them: $g_2(n) = 2n - 2$ for $n \geq 4$ ([3] as the upper bound, [9] as the lower bound); $g_d(n) = O(n^{2d-2})$ [13] and $g_d(n) = \Omega(n^{d-1})$ [8]. Some results deal with families with the restriction that the members of the family are disjoint translates of a convex set. Katchalski, Lewis and Liu proved [7, 8] that for such families in \mathbb{R}^2 , the maximal number of geometric permutations is 3. They also conjectured [8] that for each natural d , there is a *constant* upper bound on the number of geometric permutations for families of translates in \mathbb{R}^d (the conjectured upper bound was $\frac{(d+1)!}{2}$). However, the only known upper bound in \mathbb{R}^d is $O(n^{d-1})$ (this follows from [11]). A constant upper bound is known in a special case: for families of congruent balls in \mathbb{R}^d [10] (improved in [2]; the bound is 2 when $n \geq 9$).

We refute the conjecture mentioned above by proving that there is no constant bound for the maximal number of geometric permutations for families of disjoint translates of a general convex set:

Theorem 1 *For each $n \in \mathbb{N}$, $n > 1$, there exists a convex set $X = X(n)$ in \mathbb{R}^3 and a family $\mathcal{F} = \mathcal{F}(n)$ of $2n$ disjoint translates of X that admits at least $n + 1$ geometric permutations.*

In Section 3 we show that a stronger result holds: there is a set Y that does not depend on n and satisfies Theorem 1.

A motivation for studying geometric permutations is Helly-type problems on the existence of common transversals for families of disjoint translates of a convex set. For example, geometric permutations were used in Tverberg's solution [12] of Grünbaum's conjecture: if in a family of disjoint translates of a convex set in \mathbb{R}^2 each 5 sets have a transversal line, then the whole family has a transversal. Geometric permutations were also used in the proofs of Helly-type theorems on transversals for disjoint unit balls in \mathbb{R}^3 [1, 5].

Holmsen and Matoušek showed [6] that in \mathbb{R}^3 there is no Helly-type theorem analogous to Tverberg's result mentioned above (the proof of Grünbaum's conjecture). For each $n \in \mathbb{N}$ they construct a family of disjoint translates of a convex set such that each n members of the family have a transversal line, while the entire family does not. In general, the idea of their construction is to take first a family of disjoint sets that have the desired transversal properties, but are not translates of each other, and then to append them one to another in order to obtain translates, preserving their disjointness and transversal properties. Our construction also uses this idea. Both constructions use the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. This surface has been used earlier for the construction of several examples with transversal lines (see, for example, Theorem 2.9 by Aronov, Goodman, Pollack and Wenger, in [4], p. 171, and the construction described there).

2 The construction

Let $n \in \mathbb{N}$, $n > 1$.

Planes and Lines

Denote by Σ the hyperbolic paraboloid $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = xy\}$. For each $i \in \{0, 1, \dots, n\}$, let λ_i be the plane $y = i$, and let l_i be the line $\lambda_i \cap \Sigma = \{(x, y, z) : y = i, z = xi\}$. These $n + 1$ lines will be transversal lines for \mathcal{F} , inducing different geometric permutations.

For each $m \in \{1, 2, \dots, n\}$, let u_m denote the plane $x = 2mn^2$, u'_m the plane $x = 2mn^2 + 1$, and w_m the plane $x = 2mn^2 + n^2 + 1/2$. These planes will be used in the construction of \mathcal{F} , and in the proof of the disjointness of its members.

The set X

For each $m \in \{1, 2, \dots, n\}$, define four points on Σ as follows:

$$\begin{aligned} P_{m,1} &= l_{m-1} \cap u_m = (2mn^2, m-1, 2mn^2 \cdot (m-1)), \\ P_{m,2} &= l_m \cap u'_m = (2mn^2 + 1, m, (2mn^2 + 1) \cdot m); \\ Q_{m,1} &= l_m \cap u_m = (2mn^2, m, 2mn^2 \cdot m), \\ Q_{m,2} &= l_{m-1} \cap u'_m = (2mn^2 + 1, m-1, (2mn^2 + 1) \cdot (m-1)). \end{aligned}$$

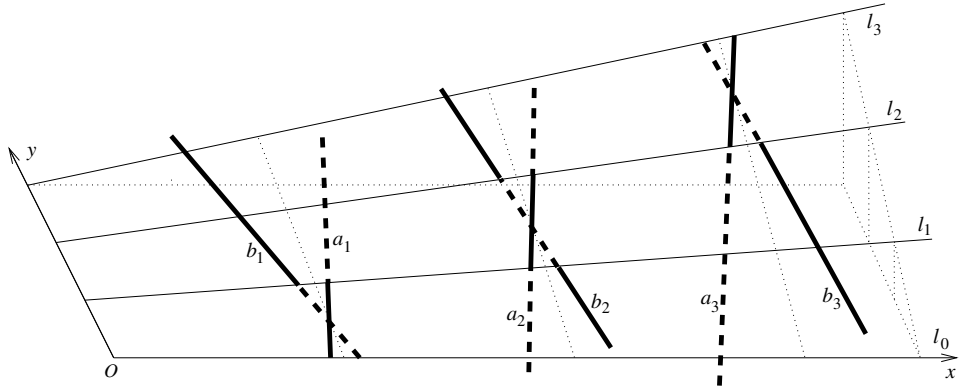


Figure 1: The segments a_i and b_i , for $n = 3$. The solid parts are above Σ , and the dashed are below it.

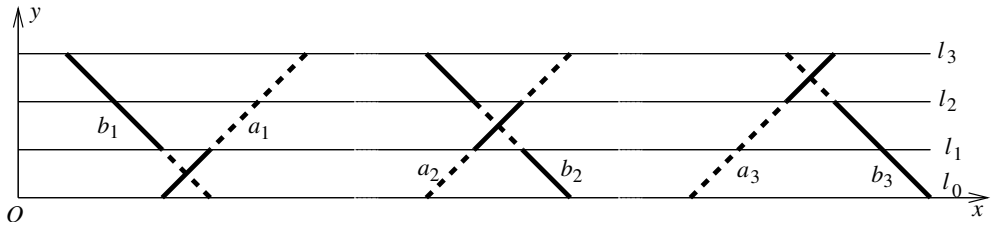


Figure 2: The segments a_i and b_i , for $n = 3$, viewed from above.

Note that $P_{m,1}, P_{m,2} \in \pi_m$, and $Q_{m,1}, Q_{m,2} \in \sigma_m$, where π_m is the plane $y = x - 2mn^2 + m - 1$ and σ_m is the plane $y = -x + 2mn^2 + m$. The planes π_m and σ_m are parallel to the planes $y = x$ and $y = -x$ respectively.

Let a_m be the segment that contains $P_{m,1}$ and $P_{m,2}$ with endpoints in the planes λ_0 and λ_n , and let b_m be the segment that contains $Q_{m,1}$ and $Q_{m,2}$ with endpoints in the planes λ_0 and λ_n . Figures 1 and 2 show a_i -s and b_i -s for $n = 3$ (In these figures, the solid parts of the segments are above Σ , and the dashed are below it. Note that the figures are not drawn to scale: in fact, the segments are much further apart).

In what follows, “the highest (lowest) point” (of a set) means “the point with the maximal (minimal) z -coordinate”. (This is used only when such points are unique.)

Now define two sets X^L and X^U . Each of them is a polygonal line: $X^L = \tilde{a}_1 \cup \tilde{a}_2 \cup \dots \cup \tilde{a}_n$, $X^U = \tilde{b}_1 \cup \tilde{b}_2 \cup \dots \cup \tilde{b}_n$, where each \tilde{a}_m is a translate of a_m , and each \tilde{b}_i is a translate of b_i , so that:

- the lowest point of \tilde{a}_1 is $(0, 0, 0)$, and for each $m \in \{2, 3, \dots, n\}$ the lowest point of \tilde{a}_m coincides with the highest point of \tilde{a}_{m-1} ;
- the highest point of \tilde{b}_1 is $(0, n^2, H_L + H + H_U)$ (where H_L and H_U are the z -heights of X^L and X^U respectively, and H is a large positive number), and for each $m \in \{2, 3, \dots, n\}$ the highest point of \tilde{b}_m coincides with the lowest point of \tilde{b}_{m-1} .

We make some observations on X^L and X^U .

1. The polygonal line X^L lies in the plane $y = x$ (this is true since each a_m lies in π_m which is a vertical plane parallel to $y = x$, see Figure 2; and \tilde{a}_1 contains the point $(0, 0, 0)$). Similarly, X^U lies in the plane $y = -x + n^2$.
2. Each a_m contributes n to the lengths of the x - and y -projections of X^L , and each b_m contributes n to the lengths of the x - and y -projections of X^U . Thus the x - and y -projections of X^L and of X^U are $[0, n^2]$.
3. The slopes of a_m and b_m , relative to the plane $z = 0$, are, respectively, $\frac{1}{\sqrt{2}}((2mn^2 + 1)m - 2mn^2(m - 1)) = \frac{1}{\sqrt{2}}(2n^2 + 1)m$ and $\frac{1}{\sqrt{2}}((2mn^2 + 1)(m - 1) - 2m^2n^2) = -\frac{1}{\sqrt{2}}((2n^2 - 1)m + 1)$. This means that if $m < m'$ then the slope of a_m is smaller than that of $a_{m'}$, and the slope of b_m is smaller than that of $b_{m'}$. It follows that X^L is downward convex, and X^U is upward convex polygonal line.

Let $X = \text{conv}(X^L \cup X^U)$ (see Figure 3; note that in fact X^U is situated high above X^L).

The family \mathcal{F} of disjoint translates of X

For each $m \in \{1, 2, \dots, n\}$, define A_m to be a translate of X with \tilde{a}_m translated to a_m , and B_m to be a translate of X with \tilde{b}_m translated to b_m . Denote by A_m^U (A_m^L) the polygonal line on A_m that corresponds to X^U (X^L) on X , and by B_m^U (B_m^L) the polygonal line on B_m that corresponds to X^U (X^L) on X .

The family $\mathcal{F} = \{A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n\}$ is a family of $2n$ translates of X . We prove that they are pairwise disjoint, and that \mathcal{F} has at least $n + 1$ geometric permutations.

Disjointness of the members of \mathcal{F}

First, note that for each $m \in \{1, 2, \dots, n - 1\}$, the sets A_m and B_m have points $(P_{m,1}$ and $Q_{m,1}$ respectively) with x -coordinate $2mn^2$, and the sets A_{m+1} and B_{m+1} have points $(P_{m+1,2}$ and $Q_{m+1,2}$ respectively) with x -coordinate $2(m + 1)n^2 + 1$. The x -lengths of the sets are n^2 . It follows that A_m and B_m are to the left of the plane w_m (recall that this plane is $x = 2mn^2 + n^2 + 1/2$), while A_{m+1} and B_{m+1} are to its right. Hence for $m \neq m'$, $A_m \cap A_{m'}$, $A_m \cap B_{m'}$ and $B_m \cap B_{m'}$ are \emptyset .

It remains to prove that $A_m \cap B_m = \emptyset$ for each $m \in \{1, 2, \dots, n\}$. Let τ_m be the plane that contains the point $(2mn^2 + 1/2, m - 1/2, (2mn^2 + 1/2)(m - 1/2))$, and parallel to a_m and b_m . We claim that this plane separates A_m from B_m .

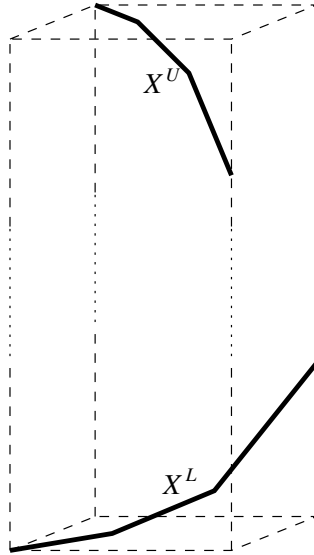


Figure 3: The set X for $n = 3$: the bold polygonal lines are X^L and X^U ; X is their convex hull.

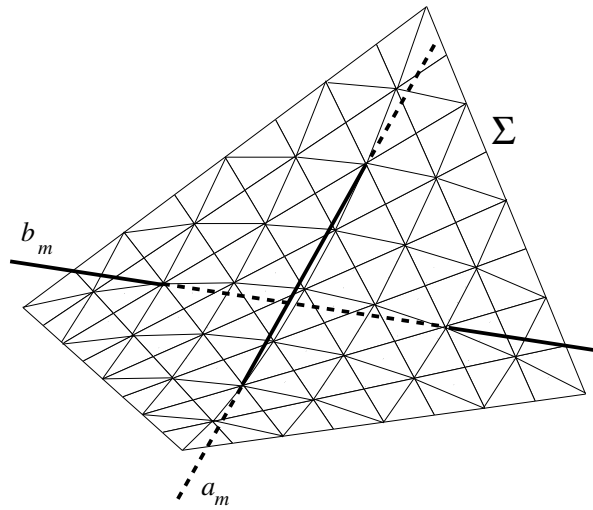


Figure 4: Illustration to the proof of the disjointness of A_m and B_m : the segment a_m is above b_m .

Let $t_m = \pi_m \cap \tau_m$, $s_m = \sigma_m \cap \tau_m$ (the planes π_m and σ_m were defined in the beginning of this section). The segment a_m is parallel to τ_m and lies in π_m , hence a_m is parallel to t_m , and it is easy to check that a_m is above t_m in the vertical plane π_m . We have observed that X^L is downward convex. Hence A_m^L , being a translate of X^L , is also above t_m , and thus above τ_m . Similarly, the segment b_m is parallel to s_m and is below it in the vertical plane σ_m . We have observed that X^U is upward convex. Hence B_m^U , being a translate of X^U , is also below s_m , and thus above τ_m (see Figure 4).

If H in the definition of X_U is large enough, then also A_m^U is above τ_m , and B_m^L is below τ_m . Since $A_m = \text{conv}(A_m^L \cup A_m^U)$ and $B_m = \text{conv}(B_m^L \cup B_m^U)$, it follows that A_m is above τ_m , and B_m is below τ_m . Hence $A_m \cap B_m = \emptyset$.

Transversal properties

We prove that each A_m and each B_m meets each l_i .

- For A_m :

For $i = m - 1$, $P_{m,1} = l_{m-1} \cap a_m$; for $i = m$, $P_{m,2} = l_m \cap a_m$.

For each $i \neq m - 1, m$: since the curve $\Sigma \cap \pi_m$ is downward convex, the point $\lambda_i \cap a_m$, which belongs to A_m^L , lies below Σ and hence below l_i . Since the projection of A_m^U on the y -axis is equal to the projection of A_m^L , there is a point of A_m^U that belongs to λ_i . Since A_m^U is high above A_m^L , this point is above l_i , and it follows that A_m meets l_i .

- For B_m :

For $i = m - 1$, $Q_{m,2} = l_{m-1} \cap b_m$; for $i = m$, $Q_{m,1} = l_m \cap b_m$.

For each $i \neq m - 1, m$: since the curve $\Sigma \cap \sigma_m$ is upward convex, the point $\lambda_i \cap b_m$, which belongs to B_m^U , lies above Σ and hence above l_i . Since the projection of B_m^L on the y -axis is equal to the projection of B_m^U , there is a point of B_m^L that belongs to λ_i . Since B_m^U is high above B_m^L , this point is below l_i , and it follows that B_m meets l_i .

Geometric permutations

Let T_m^A and T_m^B be the open halfspaces bounded by τ_m , that contain A_m and B_m respectively. Let $O_i = (0, i, 0) \in l_i$. On each l_i , choose direction such that O_i is before all the points with positive x -coordinate. It follows that for each m , on each l_i , $O_i \prec A_m$ and $O_i \prec B_m$. Note also that for $m < m'$, on each l_i we have $A_m \prec A_{m'}$, $A_m \prec B_{m'}$, $B_m \prec A_{m'}$ and $B_m \prec B_{m'}$, since the planes w_m separate such pairs of sets. However, the order of A_m and B_m on l_i depends on i :

- For $i = m - 1$: on l_{m-1} , $O_{m-1} \prec P_{m,1} \prec Q_{m,2}$, and thus $A_m \prec B_m$. Note that $O_{m-1} \in T_m^A$.
- For $i = m$: on l_m , $O_m \prec Q_{m,1} \prec P_{m,2}$, and thus $B_m \prec A_m$. Note that $O_m \in T_m^B$.
- For any i : we have observed that $O_{m-1} \in T_m^A$ and $O_m \in T_m^B$. It follows that $O_i \in T_m^A$ for $i \leq m - 1$, and $O_i \in T_m^B$ for $i \geq m$. Since both A_m

and B_m meet each l_i after O_i , we have $A_m \prec B_m$ on l_i for $i \leq m-1$ and $B_m \prec A_m$ on l_i for $i \geq m$.

We obtain the following geometric permutations for \mathcal{F} :

$$\begin{aligned} l_0 &: (A_1, B_1, A_2, B_2, A_3, B_3, \dots, A_n, B_n) \\ l_1 &: (B_1, A_1, A_2, B_2, A_3, B_3, \dots, A_n, B_n) \\ l_2 &: (B_1, A_1, B_2, A_2, A_3, B_3, \dots, A_n, B_n) \\ l_3 &: (B_1, A_1, B_2, A_2, B_3, A_3, \dots, A_n, B_n) \\ &\dots \\ l_n &: (B_1, A_1, B_2, A_2, B_3, A_3, \dots, B_n, A_n). \end{aligned}$$

Thus \mathcal{F} is a family of $2n$ disjoint translates of the convex set X that has the $n+1$ geometric permutations listed above (they are distinct if $n > 1$).

3 Concluding remarks

1. The set $X = X(n)$ that has been constructed in Section 2, depends on n . Here we explain, leaving some details to the reader, how to construct a set that satisfies Theorem 1 for all values of $n \in \mathbb{N}$.

Recall the construction of $X(n)$. It is easy to see that it is possible to modify the construction so that λ_i is the plane $y = i\epsilon$ for some positive constant $\epsilon \leq 1$; $l_i = \lambda_i \cap \Sigma$; all the segments a_m and b_m are still parallel to the planes $y = x$ and $y = -x$ respectively, but have x - and y -lengths $n\epsilon$ (then the x - and y -lengths of X are $n^2\epsilon$). Choosing $\epsilon \leq 1/n^2$, it is possible to modify the construction so that the planes $x = 5(2m-1)$ and $x = 5(2m+1) + \epsilon$ play the roles of u_m and u'_m (respectively) in the definition of the points $P_{m,j}$ and $Q_{m,j}$, and the planes $x = 10m$ play the role of w_m in the proof of the disjointness of the translates (note that in this case the x - and y -lengths of X are less than 1). Once this is done, it is possible to “squeeze” the construction (applying the transformation $(x, y, z) \mapsto (x, y, \delta z)$ for a constant $0 < \delta \leq 1$) so that the slopes of all a_m ’s and b_m ’s will be positive but less than a constant α .

Using these observations, we construct a set Y that satisfies Theorem 1 for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, construct *modified* $X^L(n)$ and $X^U(n)$, and *modified* $X(n) = \text{conv}(X^L(n) \cup X^U(n))$, so that:

1. The x - and y -lengths of $X^L(n)$ and $X^U(n)$ are $1/2^n$;
2. The slopes of all $a_m(n)$ ’s and $b_m(n)$ ’s are positive but less than the slopes of all $a_m(n-1)$ ’s and $b_m(n-1)$ ’s, and less than $1/2^n$.

Append $X^L(n)$ ’s ($X^U(n)$ ’s) in order to obtain a polygonal line Y^L (Y^U) in the way similar to the joining of the segments a_m (b_m) in the construction of X^L (X^U). That is, let $Y^L = \bigcup_{n=1}^{\infty} \tilde{X}^L(n)$ and $Y^U = \bigcup_{n=1}^{\infty} \tilde{X}^U(n)$ where $\tilde{X}^L(n)$ ($\tilde{X}^U(n)$) is a translate of *modified* $X^L(n)$ ($X^U(n)$), and the lowest point of $\tilde{X}^L(n)$ coincides with the highest point of $\tilde{X}^L(n+1)$ (the highest point of $\tilde{X}^U(n)$ coincides with the lowest point of $\tilde{X}^U(n+1)$). Because of the conditions 1 and 2 above, the sequences of the lowest points of $\tilde{X}^L(n)$ and of the highest

points of $\tilde{X}^U(n)$ converge, and the polygonal lines Y^L and Y^U have x - and y -lengths 1 ($= \sum_{n \in \mathbb{N}} 1/2^n$), and finite z -lengths. It remains to put Y^U high above Y^L (so that they have the same x - and y -projections, say, $[0, 1]$), and to define $Y = \text{conv}(Y^L \cup Y^U)$. The set Y looks similar to the set from Figure 2, but the bold polygonal lines consist of an infinite number of segments.

For each natural n , it is possible to place $2n$ translates of Y so that the segments of Y^L (Y^U) that correspond to $a_m(n)$'s ($b_m(n)$'s) coincide with these segments in $2n$ translates of the *modified* $X(n)$. This translates are disjoint, and they have the mentioned above $n + 1$ geometric permutations. This proves our statement.

2. Theorem 1 shows that for $d \geq 3$, for families of disjoint translates of a convex set, $g_d(n) = \Omega(n)$. We think that it is possible, using similar constructions, to improve this bound for $d > 3$.

3. To summarize, for families of disjoint translates of a convex set, $g_3(n) = O(n^2)$ and $\Omega(n)$. The problem of narrowing this gap remains open.

References

- [1] O. Cheong, X. Goaoc, and A. Holmsen. Hadwiger and Helly-type theorems for disjoint unit spheres in \mathbb{R}^3 . In *Proc. of ACM Symposium on Computational Geometry*, 2005; invited to the special issue of *Computational Geometry: Theory and Applications* on SoCG, 2005.
- [2] O. Cheong, X. Goaoc, and H.-S. Na. Geometric permutations of disjoint unit spheres, *Comput. Geom.: Theory and Applications* **30** (2005), 253-270.
- [3] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex nonintersecting sets in the plane is $2n - 2$, *Discrete Comput. Geom.* **5** (1990), 35-42.
- [4] J.E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In *New Trends in Discrete and Computational Geometry*, J. Pach, ed., vol. 10 of Algorithms and Combinatorics. Springer-Verlag, Heidelberg, Germany, 1993, 163-198.
- [5] A. Holmsen, M. Katchalski, and T. Lewis. A Helly-Type theorem for line transversals to disjoint unit balls, *Discrete Comput. Geom.* **29** (2003), 595-602.
- [6] A. Holmsen and J. Matoušek. No Helly theorem for stabbing translates by lines in \mathbb{R}^3 , *Discrete Comput. Geom.* **31(3)** (2004), 405-410.
- [7] M. Katchalski, T. Lewis, and A. Liu. Geometric permutations of disjoint translates of convex sets, *Discrete Math.* **65** (1987), 249-260.
- [8] M. Katchalski, T. Lewis, and A. Liu. The different ways of stabbing disjoint sets, *Discrete Comput. Geom.* **7** (1992), 197-206.

- [9] M. Katchalski, T. Lewis, and J. Zaks. Geometric permutations for convex sets, *Discrete Math.* **54** (1985), 271-284.
- [10] M. Katchalski, S. Suri, and Y. Zhou. A constant bound for geometric permutations of disjoint unit balls, *Discrete Comput. Geom.* **29** (2003), 161-173.
- [11] M.J. Katz and K.R. Varadarajan. A tight bound on the number of geometric permutations of convex fat objects in \mathbb{R}^d , *Discrete Comput. Geom.* **26** (2001), 543-548.
- [12] H. Tverberg. Proof of Grünbaum's conjecture on common transversals for translates, *Discrete Comput. Geom.* **4** (1989), 191-203.
- [13] R. Wenger. Upper bounds on geometric permutations for convex sets, *Discrete Comput. Geom.* **5** (1990), 27-33.

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