Geometric Permutations for Planar Families of Disjoint Translates of a Convex Set

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Geometric Permutations for Planar Families of Disjoint Translates of a Convex Set

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Abstract

This work deals with geometric permutations of families of convex sets, mostly of planar families of pairwise disjoint translates of a convex set.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^d$. A straight line l is a *transversal* of $\mathcal{A}$ if it intersects every set in $\mathcal{A}$. Each transversal intersects the members of $\mathcal{A}$ in an order which can be described by a pair of permutations of $\{1, 2, \ldots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation.*

T-family is a finite family of pairwise disjoint translates of a convex compact set. Katchalski, Lewis and Liu proved that:

- Any T-family admits at most 3 geometric permutations.
- All geometric permutations of a T-family have representatives that coincide in all except possibly 4 consecutive places.

We extend and refine this result in several directions. The results obtained are:

- Let S be a convex polygon. There exists a minimal integer $n(S)$ such that any family of pairwise disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations.
- Let S be a strictly convex smooth set. There exists a minimal integer $n(S)$ such that any family of pairwise disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations, and they have representatives that coincide in all except possibly two consecutive places.

Counterexamples are constructed:

For any integer $n \geq 3$ there exists a convex quadrilateral $P = P(n)$ such that it is possible to construct a family of n pairwise disjoint translates of P that admits 3 geometric permutations.

For any integer $n \geq 3$ there exists a strictly convex smooth set $Q = Q(n)$ such that it is possible to construct a family of n pairwise disjoint translates of Q that admits 3 geometric permutations.

There exists a convex set M such that for any integer $n \geq 3$ it is possible to construct a family of n pairwise disjoint translates of M that admits 3 geometric permutations.

- Let S be a convex set that is not a segment. For each $n \geq 3$ denote by $P_n(S)$ the maximal number of geometric permutations that a family of n pairwise disjoint translates of S can admit. Then one of three cases occurs:

  1. for each $n \geq 3$, $P_n(S) = 2$;
  2. for each $n \geq 3$, $P_n(S) = 3$;
  3. there exists n_0 such that for $3 \leq n \leq n_0$, $P_n(S) = 3$ and for $n > n_0$, $P_n(S) = 2$.

- Any family of size ≥ 4 of disjoint congruent discs admits at most 2 geometric permutations, and they have representatives that coincide in all except possibly 2 consecutive places.

- Let $\mathcal{A}$ be a T-family of size ≥ 4 that admits 3 geometric permutations. Then the triple of geometric permutations is one of the following:

  1. $\{< W123W' >, < W213W' >, < W132W' >\}$,
  2. $\{< W1234W' >, < W2134W' >, < W2413W' >\}$.

Chapter 1

Introduction

This work deals with geometric permutations of families of convex sets.

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of pairwise disjoint convex sets in $\mathbb{R}^d$. A straight line l is a *transversal* of $\mathcal{A}$ if it intersects every set in $\mathcal{A}$. Each transversal intersects the members of $\mathcal{A}$ in an order which can be described by a pair of permutations of $\{1, 2, \ldots, n\}$ which are reverses of each other. Such a pair is called a *geometric permutation*, and each of two permutations that form it is called a *representative of geometric permutation*. Figure 1.1 shows an example of a planar family of congruent discs that admits three geometric permutations.

In this work we deal only with geometric permutations of planar families. We will use the following notation: *T-family* will denote a finite family of pairwise disjoint translates of a convex set in $\mathbb{R}^2$. All sets are assumed to be compact, and whenever we say "disjoint translates", we mean pairwise disjoint translates.

Geometric permutations were introduced by Katchalski, Lewis and Liu [10] as a tool in dealing with problems in Geometric Transversal Theory. A family $\mathcal{A}$ has *property* $\mathbf{T}_r$ if each subfamily of $\mathcal{A}$ of size $\leq r$ has a transversal; $\mathcal{A}$ has *property* $\mathbf{T}$ if the entire family has a transversal. Grünbaum conjectured [5] that for T-families $\mathbf{T}_5 \Rightarrow \mathbf{T}$ holds. A major step towards this result was done by Katchalski who proved that for T-families $\mathbf{T}_{128} \Rightarrow \mathbf{T}$, and the Grünbaum's conjecture was finally proved by Tverberg [16]. Both proofs used geometric permutations.

Results of this type can be considered as Helly-type theorems. Consult [1, 2, 4, 6, 18, 20] for other generalizations and related results.

Several results on geometric permutations were obtained [3, 10, 11, 12, 13,

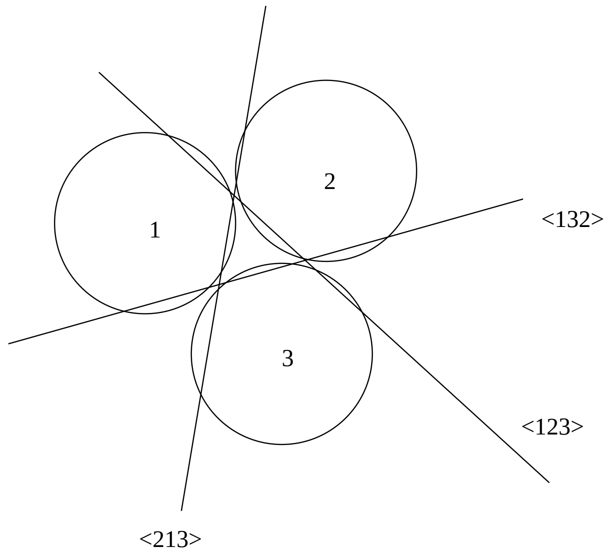


Figure 1.1: A family of congruent discs that admits three geometric permutations.

14, 15, 19]. In the planar case some exact bounds were found. Edelsbrunner and Sharir proved [3] that any family of n disjoint convex sets admits at most $2n-2$ geometric permutations. Katchalski, Lewis and Liu proved [11]:

Theorem 1.1

1. *Any T-family admits at most 3 geometric permutations.*

2. *All geometric permutations of a T-family have representatives that coincide in all except possibly 4 consecutive places.*

In this work we extend and refine this theorem in several directions.

In Chapter 2 we prove the following theorem about polygons and strictly convex smooth sets (a convex set is *smooth* if at each boundary point it has exactly one supporting line; it is *strictly convex* if its boundary contains no segments):

Theorem 2.1

1. *Let S be a convex polygon. There exists a minimal integer $n(S)$ such that any family of disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations.*

2. *Let S be a strictly convex smooth set. There exists a minimal integer $n(S)$ such that any family of disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations, and they have representatives that coincide in all except possibly 2 consecutive places.*

We construct examples that show that the numbers $n(S)$ in Theorem 2.1 are not bounded: for any integer $n \geq 3$ there exists a convex quadrilateral $P = P(n)$ such that it is possible to construct a family of n disjoint translates of P that admits 3 geometric permutations (Example 2.7), and for any integer $n \geq 3$ there exists a strictly convex smooth set $Q = Q(n)$ such that it is possible to construct a family of n disjoint translates of Q that admits 3 geometric permutations (Example 2.8).

We also construct a convex set M such that for each integer $n \geq 3$ it is possible to construct a family of n disjoint translates of M that admits 3 geometric permutations (Example 2.9).

All possible cases are summarized in the following theorem:

Theorem 2.2 *Let S be a convex set that is not a segment. For each $n \geq 3$ denote by $P_n(S)$ the maximal number of geometric permutations that a family of n disjoint translates of S can admit. Then one of three cases occurs:*

1. *for each $n \geq 3$, $P_n(S) = 2$;*
2. *for each $n \geq 3$, $P_n(S) = 3$;*
3. *there exists n_0 such that for $3 \leq n \leq n_0$, $P_n(S) = 3$ and for $n > n_0$, $P_n(S) = 2$.*

In Chapter 3 we describe all possible geometric permutations of families of disjoint translates of a disc:

Theorem 3.1 *Any family of size ≥ 4 of disjoint congruent discs admits at most 2 geometric permutations, and they have representatives that coincide in all except possibly 2 consecutive places.*

A weaker version of Theorem 3.1 was obtained by Smorodinsky, Mitchell and Sharir [14, 15].

In Chapter 4 we find all compatible triples of geometric permutations of T-families of size ≥ 4:

Theorem 4.1 *Let $\mathcal{A}$ be a T-family of size ≥ 4 that admits 3 geometric permutations. Then the triple of geometric permutations in one of the following:*

1. $\{< W123W' >, < W213W' >, < W132W' >\}$,

2. $\{< W1234W' >, < W2134W' >, < W2413W' >\}$.

Theorem 4.1 is a refinement of a result obtained by Tverberg [17].

In Chapter 5 we mention some open problems related to geometric permutations.

Chapter 2

Polygons and Strictly Convex Smooth Sets

Theorem 1.1 asserts that each T-family admits at most 3 geometric permutations. In this chapter we show that in some cases 3 can be replaced by 2. The cases are that of "sufficiently many translates of a convex polygon" and that of "sufficiently many translates of a strictly convex smooth set".

A convex set is *strictly convex* if its boundary contains no segment; it is *smooth* if it has exactely one supporting line at each point of its boundary.

Theorem 2.1

1. *Let S be a convex polygon. There exists a minimal integer $n(S)$ such that any family of disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations.*

2. *Let S be a strictly convex smooth set. There exists a minimal integer $n(S)$ such that any family of disjoint translates of S of size $n > n(S)$ admits at most 2 geometric permutations, and they have representatives that coincide in all except two consecutive places.*

We construct examples that show that the numbers $n(S)$ from Theorem 2.1 are not bounded (Examples 2.7 and 2.8).

We also give an example of a set M such that for each integer n it is possible to construct a family of disjoint translates of M that admits 3 geometric permutations(Example 2.9).

A special case of Theorem 2.1 when S is a disc was proved by Smorodinsky, Mitchell and Sharir [14, 15]. In Chapter 3 we prove that in this case $n(S) = 3$.

The final result in this section is:

Theorem 2.2 *Let S be a convex set that is not a segment. For each $n \geq 3$ denote by $P_n(S)$ the maximal number of geometric permutations that a family of n disjoint translates of S can admit. Then one of three cases occurs:*

1. *for each $n \geq 3$, $P_n(S) = 2$;*
2. *for each $n \geq 3$, $P_n(S) = 3$;*
3. *there is n_0 such that for $3 \leq n \leq n_0$, $P_n(S) = 3$ and for $n > n_0$, $P_n(S) = 2$.*

2.1 Definitions and Notations

Whenever we consider two directed lines l_1 and l_2, we assign the numerals I, II, III, IV to the four quadrants formed by l_1 and l_2 (the lines are not necessarily orthogonal), as in Figure 2.1, and denote by O the point of intersection of l_1 and l_2.

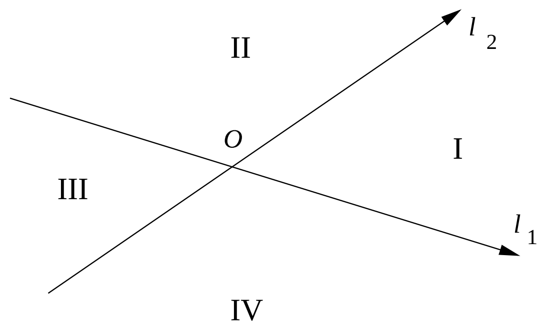


Figure 2.1: Four quadrants formed by l_1 and l_2.

We say that a set *crosses* some quadrant if it intersects both rays that form this quadrant but does not contain O (See Figure 2.2).

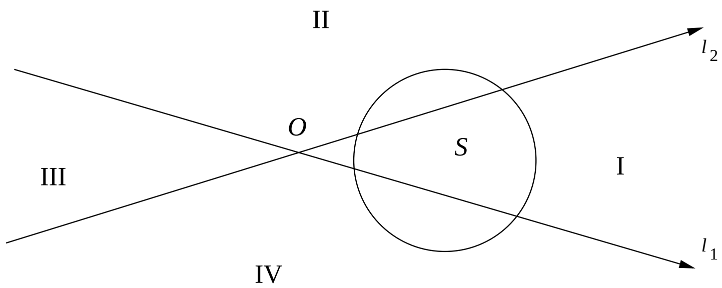


Figure 2.2: Disc S crosses quadrant I.

2.2 Proof of Theorem 2.1 for polygons

We begin by proving two lemmas.

Lemma 2.3 *Let S be a convex set with positive area. For each $\varepsilon > 0$ there exists an integer $n = n(\varepsilon, S)$ such that if $\mathcal{A}$ is a family of n disjoint translates of S, and l_1 and l_2 are two transversals of $\mathcal{A}$, then the small angle between l_1 and l_2 is less than ε.*

(A similar lemma was proved by Smorodinsky, Mitchell and Sharir [14, 15].)

Proof: Assume to the contrary that there exists an $\varepsilon > 0$ such that for each n there exists a family of n disjoint translates of S that has two transversals l_1 and l_2 such that the angle between l_1 and l_2 is greater than ε.

Consider a strip extending $diam(S)$ in each direction from l_1, and a strip extending $diam(S)$ in each direction from l_2. All the members of the family are contained in the intersection of the strips. But the intersection of the strips is a parallelogram whose area is bounded by $\frac{(2diam(S))^2}{\sin\varepsilon}$ and hence it can contain no more than $\frac{(2diam(S))^2}{(\sin\varepsilon)(area(S))}$ disjoint translates of S.
■

Observation 2.4 *Let S be a convex polygon. Let $\mathcal{A}$ be a family of disjoint translates of S, and l_1 and l_2 transversals of $\mathcal{A}$ intersecting in O, such that l_1 and l_2 induce different geometric permutations. Then S has an edge which is parallel to a line m_1 that contains O and belongs to the union of the odd quadrants, and another edge which is parallel to a line m_2 that contains O*

and belongs to the union of the even quadrants, and the lines m_1 and m_2 are different from l_1 and l_2.

(See Figure 2.3.)

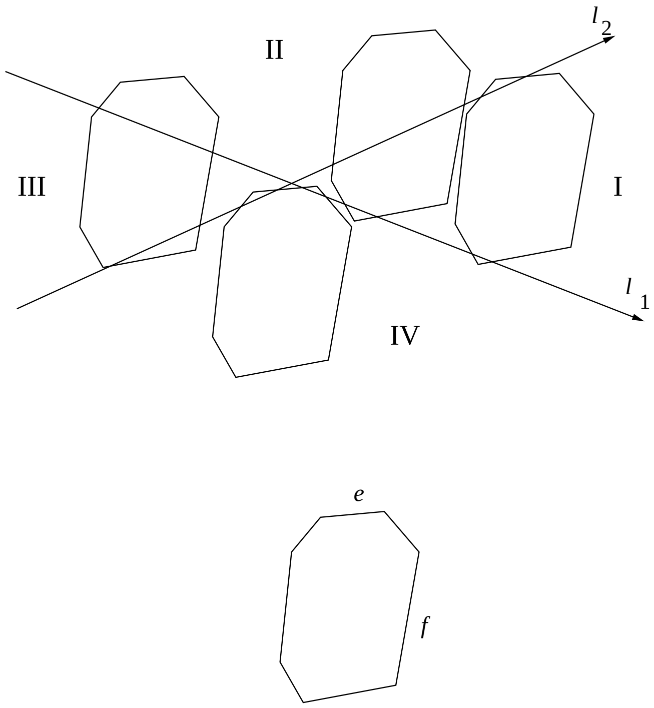


Figure 2.3: The edge e is parallel to a line that contains O and belongs to the union of the odd quadrants; the edge f is parallel to a line that contains O and belongs to the union of the even quadrants.

Proof: It suffices to prove that S has an edge which is parallel to a line that contains O, belongs to the union of odd quadrants and is different from l_1 and l_2.

Assume that a polygon S crosses a certain quadrant, say IV. Then a line l containing an edge of S separates S from O. This line crosses quadrant IV (because it separates $S \bigcap l_1$ and $S \bigcap l_2$ from O) and its translation $\tilde{l}$ to O belongs to the union of the odd quadrants, and is different from l_1 and l_2 (See Figure 2.4).
■

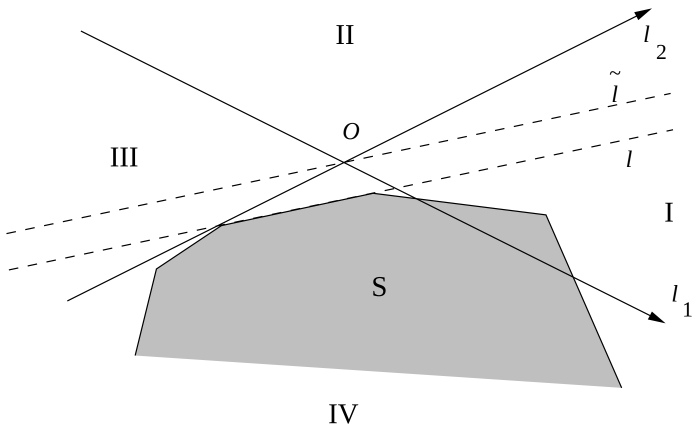


Figure 2.4: Illustration to the proof of Observation 2.4.

Proof of Theorem 2.1 for polygons: Suppose that a family $\mathcal{A}$ of disjoint translates of S has three transversals l_1, l_2, l_3 that induce different geometric permutations. Then, by Lemma 2.3, for each ε there exists an integer $n = n(\varepsilon, S)$ such that if $|\mathcal{A}| > n$ then the angle between l_1 and l_2 and the angle between l_2 and l_3 are less than ε. It can be assumed that l_1 is horizontal, l_2 has positive slope less than ε, l_3 has negative slope greater than $-\varepsilon$. Then, by Observation 2.4, S has an edge e_1 with positive slope less than ε, and also an edge e_2 with negative slope greater than $-\varepsilon$. It means that S has two nonparallel edges such that absolute value of the difference of their slopes is less than 2ε. Since this is true for each ε, we obtain a contradiction to the fact that S is given *a priori*.
■

2.3 Proof of Theorem 2.1 for strictly convex smooth sets

Two lemmas are needed for the proof.

Lemma 2.5 *Let S be a strictly convex set, and $\angle AOB$ a varying angle such that both rays OA and OB of $\angle AOB$ intersect S (we assume that A (B) is the point of $OA \bigcap S$ ($OB \bigcap S$) closest to O). Then $|OA| \to 0$ and $|OB| \to 0$ as $\angle AOB \to 180°$.*

Proof: Consider a sequence of angles $\angle A_nO_nB_n$ such that for each n, $\angle A_nO_nB_n > 180° - 1/n$. For sufficiently large n, O_n must be close to S; A_n and B_n belong to S for each n, hence these sequences are bounded. Hence we can assume, by taking subsequences, that the sequences $\{O_n\}_{n=1}^{\infty}$, $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ converge to O, A, B respectively. Now O, A, B belong to S. If O belongs to the interior of S, there is nothing to prove. Otherwise, O belongs to the boundary of S, and A, B belong to lines which support S at O. Hence $A = B = O$, since otherwise the boundary of S contains the segment OA or the segment OB contradicting the assumption that S is a strictly convex set.
■

Lemma 2.6 *Let S be a smooth convex set. Then there exists an $\varepsilon = \varepsilon(S) > 0$ such that for any X, Y, Z which are disjoint translates of S, and for each point p of the plane at least one of $dist(p, X)$, $dist(p, Y)$, $dist(p, Z)$ is greater than ε.*

Proof: Assume to the contrary that for each ε there are X, Y and Z which are disjoint translates of S and a point p such that all of $dist(p, X)$, $dist(p, Y)$ and $dist(p, Z)$ are less than ε. It is possible to assume that p is a constant point. Then for each integer n there exist points x_n, y_n, z_n such that $S + x_n$, $S + y_n$, $S + z_n$ are disjoint and all of $dist(p, S + x_n)$, $dist(p, S + y_n)$, $dist(p, S + z_n)$ are less than $1/n$.

The sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are bounded since S is bounded, thus it can be assumed, by taking subsequences, that they converge to x, y and z respectively. The point p belongs to the boundaries of $S+x$, $S+y$ and $S+z$, and no two of these sets have common interior points.

For each n, $S + x_n$ and $S + y_n$ can be separated by a line. Thus $S + x$ and $S + y$ can be separated by a line l. The line l contains p since this point is common to the two sets and l separates them.

The same is true for the sets $S + x$ and $S + z$. Since S is a smooth set, there is only one line supporting $S + x$ at P. Thus, $S + y$ and $S + z$ lie on the same side of l. Since they have no common interior points, they can be separated by a line $m \neq l$. The line m, as well as l, contains p, thus $S + y$ has two supporting lines at p, contradicting the fact that S is smooth.
■

Proof of Theorem 2.1 for strictly convex smooth sets: Let $\mathcal{A}$ be a family of disjoint translates of S and let l_1 and l_2 be two directed transversals of $\mathcal{A}$ intersecting at O such that l_1 and l_2 induce representatives of different geometric permutations. We prove that if $|\mathcal{A}| > n_0$ (to be defined later), then the geometric permutations induced by l_1 and l_2 have representatives that coincide in all except two consecutive places.

Let A and B be two sets in $\mathcal{A}$. We shall say that the set A *majors* B if A crosses a certain quadrant and either B contains O or for one of the rays $\vec{r}$ bounding the quadrant which A crosses: the intersection $B \bigcap \vec{r}$ is between O and $A \bigcap \vec{r}$ (see Figure 2.5). We shall say that B is *opposite* A if A and B cross opposite quadrants.

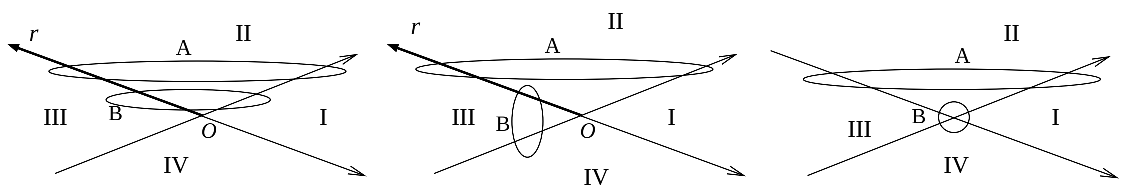


Figure 2.5: A majors B.

Let $\varepsilon = \varepsilon(S)$ be as Lemma 2.6.

Let β° be an angle such that if a translate X of S meets two rays emanating from O and forming an angle β° or more, then the distances from O to the intersections of X with the two rays are both less than ε. Such a β exists by Lemma 2.5.

Finally, let $n_0 = n_0(\beta, S)$ be an integer such that if $|\mathcal{A}| > n_0$ then the big angles formed by l_1 and l_2 are greater than β°. W.l.o.g., the quadrants formed by the big angles are the even quadrants. Such an n_0 exists by Lemma 2.3.

Let T be the collection of members X of $\mathcal{A}$ such that either X crosses an even quadrant or a member Y of $\mathcal{A}$ majors X with Y crossing an even quadrant.

If $|\mathcal{A}| > n_0$ then $|T| \leq 2$ since otherwise by Lemma 2.5 the distances of three members of $\mathcal{A}$ to O are all at most ε, contradicting Lemma 2.6.

There is at least one member of $\mathcal{A}$, say A, that crosses an even quadrant, say II, because otherwise l_1 and l_2 induce the same geometric permutation. There is at least one member of $\mathcal{A}$, say B, such that either A majors B

or B is opposite A, because otherwise l_1 and l_2 induce the same geometric permutations. Since $|T| \leq 2$, there is *exactly* one such B (see Figure 2.6.) It follows that if $|\mathcal{A}| > n_0$, then any two different geometric permutations of $\mathcal{A}$ have representatives that coincide in all except two consecutive places which represent the members of T.

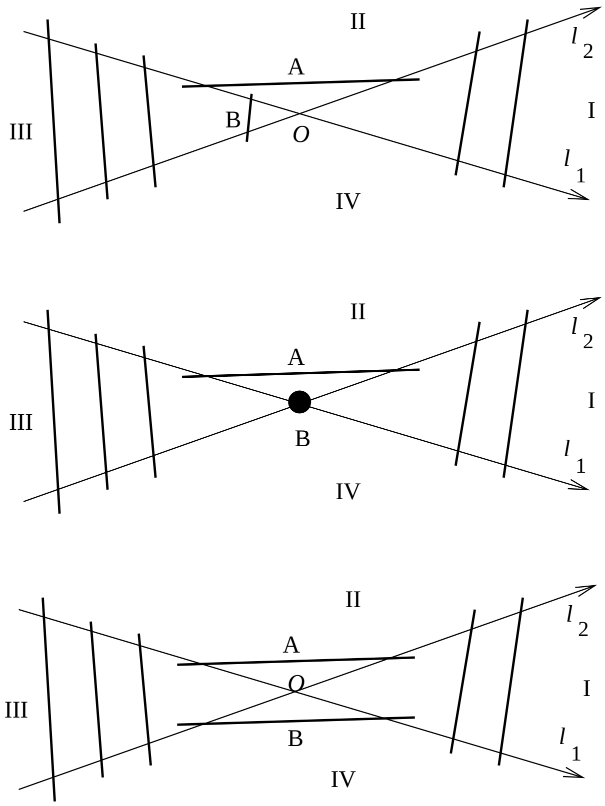


Figure 2.6: For sufficiently large n geometric permutations induced by l_1 and l_2 coincide in all except two consecutive places.

We abbreviate "coincides in all except two consecutive places" by "almost coincides".

We now prove that if $n > 3$ and any two geometric permutations of $\mathcal{A}$ have representatives that almost coincide, then $\mathcal{A}$ cannot admit 3 geometric permutations.

Assume to the contrary that $\mathcal{A}$ admits 3 geometric permutations: $\tilde{p} = \{p, -p\}$, $\tilde{q} = \{q, -q\}$, $\tilde{r} = \{r, -r\}$ (x and $-x$ are the two representatives of the geometric permutation $\tilde{x}$. Thus, if $x = (i_1, i_2, \dots, i_m)$, then $-x = (i_m, i_{m-1}, \dots, i_1))$.

Assume w.l.o.g. that p almost coincides with q and that q almost coincides with r. Then p almost coincides with r: otherwise p almost coincides with $-r$; w.l.o.g., p coincides with q in first place with set the X in the first place in both. Hence r has X in the first or in the second place (since r almost coincides with q) but r also has X in the last place or in the place before the last place (since $-r$ almost coincides with p), and this is impossible.

Assume that p does not agree with q in the i and in the $i+1$ places with the sets X and Y in the i and in the $i+1$ places of p. If r has X or Y in one of these places then r coincides with p or with q. This means that r does not have X or Y in these places and thus r disagree with p in more than two places.

It follows that if $|\mathcal{A}| > n_0$ then $\mathcal{A}$ cannot have more than 2 different geometric permutations, proving the theorem.
■

2.4 Counterxamples

We show by examples that the integers $n(S)$ in Theorem 2.1 are not bounded.

Example 2.7 Figure 2.7 presents an example of a family of four disjoint translates of a quadrilateral that admits 3 geometric permutations. For any $n > 4$, by making the vertical edges of the quadrilateral longer, it is possible to construct a family $\mathcal{A}$ of n disjoint translates of a quadrilateral such that $\mathcal{A}$ admits 3 geometric permutations.

Example 2.8 For any $n \geq 3$ it is possible by making small changes in the sets in Example 2.7 to construct a family $\mathcal{A}$ of n disjoint translates of a strictly convex smooth set such that $\mathcal{A}$ admits 3 geometric permutations.

Example 2.9 Figure 2.8 presents a set M that has the following property: for each n there exists a family of n disjoint translates of M that admits 3 geometric permutations. For any given $n \geq 3$, if we place the sets M_1 and M_3 sufficiently close, it is possible to find transversals such that the angle between them will be very small, and hence it is possible to add more sets that intersect the transversals in order to construct a family $\mathcal{A}$ of n disjoint translates of M such that $\mathcal{A}$ admits 3 geometric permutations.

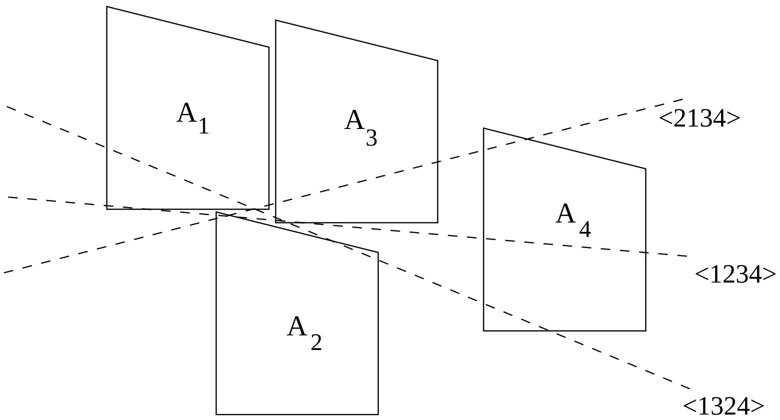


Figure 2.7: Four translates of a quadrilateral with 3 geometric permutations.

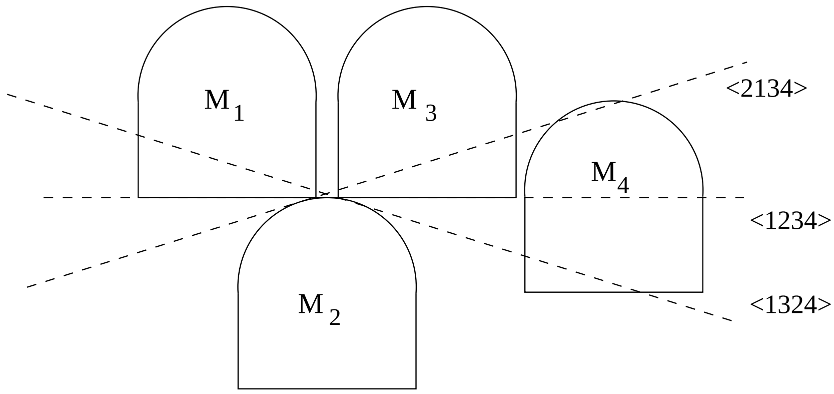


Figure 2.8: Four translates of set M with 3 geometric permutations.

2.5 Proof of Theorem 2.2

We cite some results proved by Katchalski, Lewis and Liu and by Tverberg.

Proving Theorem 1.1 in [11], Katchalski, Lewis and Liu proved the following lemma:

Lemma 2.10 *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ be a T-family. If $< 1234 >$ is a geometric permutation for $\mathcal{A}$, then neither $< 3214 >$ nor $< 2143 >$ can be geometric permutations for $\mathcal{A}$.*

We say that the pairs of geometric permutations $\{< 1234 >, < 3214 >\}$

and $\{< 1234 >, < 2143 >\}$ are *incompatible pairs* for T-families. (It should be noted that the pair $\{< 1234 >, < 2143 >\}$ is also incompatible for families of disjoint covex sets which are not necessairly translates.)

The following lemma was proved in [11] as a direct consequence of Lemma 2.10:

Lemma 2.11 *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ be a T-family. It is impossible that $\mathcal{A}$ admits two geometric permutations from the same column in the following table:*

$< 1234 >$	$< 1243 >$	$< 1324 >$
$< 2143 >$	$< 2134 >$	$< 3142 >$
$< 1432 >$	$< 1342 >$	$< 1423 >$
$< 4123 >$	$< 3124 >$	$< 4132 >$

Tverberg [17] obtained the following results:

Theorem 2.12 *If a T-family of size ≥ 4 admits 3 geometric permutations, then they are of the form $\{< WK_1W' >, < WK_2W' >, < WK_3W' >\}$ where K_1, K_2, K_3 are representatives of distinct geometric permutations of four sets.*

Observation 2.13 *Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ be a T-family. There are just four (up to relabeling) triples of geometric permutations for $\mathcal{A}$ that do not contradict Lemma 2.10. They are:*

1. $\{< 1234 >, < 1342 >, < 1423 >\}$;
2. $\{< 1234 >, < 1324 >, < 1243 >\}$;
3. $\{< 1234 >, < 4132 >, < 2431 >\}$;
4. $\{< 1234 >, < 2134 >, < 3142 >\}$.

Observation 2.13 is a direct consequence of Lemma 2.11: each triple obtained by taking one geometric permutation from each column in Lemma 2.11 is equal, up to relabeling, to one of the triples mentioned in Observation 2.13.

Proof of Theorem 2.2: If S is a convex set which is not a segment, then for any $n \geq 3$ it is possible to construct a family of n disjoint translates of

S that admits 2 geometric permutations. This follows from the fact that the slope of the tangent line is a convex function and thus has at most a countable number of discontinuities. (See Figure 2.9 for an illustration: the transversals l and m are tangent to the translates A and B of S with their points of tangency to A on two sides of a, a point of continuity of the slope of the tangent line to A. The angle between the lines l and m can be arbitrary small, hence any number of translates can be added to obtain the geometric permutations $< ABW >$ and $< BAW >$).

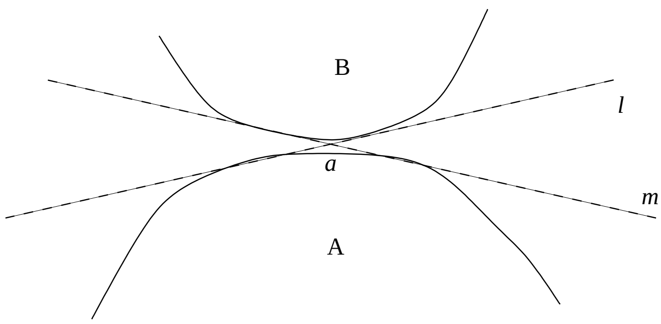


Figure 2.9: $P_n(S) \geq 2$ for $n \geq 3$.

Thus, for each $n \geq 3$, $P_n(S) \geq 2$.

On the other hand, $P_n(S) \leq 3$ by Theorem 1.1.

It remains to prove that if for some S and $n \geq 3$, $P_{n+1}(S) = 3$, then it is impossible that $P_n(S) = 2$.

Assume for some S and $n \geq 3$, $P_{n+1}(S) = 3$.

If $n = 3$ and thus $n + 1 = 4$, then there exists a family $\mathcal{A}$ of 4 disjoint translates of S that admits 3 geometric permutations. By Observation 2.13 the triple of geometric permutations is one of the following:

1. $\{< 1234 >, < 1342 >, < 1423 >\}$;

2. $\{< 1234 >, < 1324 >, < 1243 >\}$;

3. $\{< 1234 >, < 4132 >, < 2431 >\}$;

4. $\{< 1234 >, < 2134 >, < 3142 >\}$.

If the triple is of the first or of the second type, delete 1 from $\mathcal{A}$. If the triple is of the third or of the fourth type, delete 3 from $\mathcal{A}$. In all cases we obtain a T-family of size 3 admitting 3 geometric permutations.

If $n \geq 4$ and thus $n+1 \geq 5$, then there exists a family $\mathcal{A}$ of $n+1$ disjoint translates of S that admits 3 geometric permutations. By Theorem 2.12 these geometric permutations have representatives that coincide in all except four consecutive places that are representatives of distinct geometric permutations for four sets. Delete from $\mathcal{A}$ one translate that is not one of these four places. We obtain a family of n disjoint translates of S that admits 3 geometric permutations. Hence, $P_n(S) = 3$.

Thus, we proved that for any S and $n \geq 3$, $(P_{n+1}(S) = 3) \Rightarrow (P_n(S) = 3)$, and also $P_n(S) \geq 2$. Thus, the theorem is proved.

■

We proved that if S is a convex set that is not segment, then the sequence $\{P_n(S)\}_{n=3}^{\infty}$ is one of the following:

1. $2, 2, 2, \ldots;$

2. $3, 3, 3, \ldots;$

3. $3, 3, \ldots, 3, 2, 2, 2, \ldots.$

If S is a square or any parallelogram, then $\{P_n(S)\}_{n=3}^{\infty}$ is of the first type (this was proved by Katchalski, Lewis and Zaks in [13]).

If S is the set M from Example 2.9, then $\{P_n(S)\}_{n=3}^{\infty}$ is of the second type.

If S is a disc, then $\{P_n(S)\}_{n=3}^{\infty}$ is of the third type (it follows from Theorem 2.1 and from the fact that there is a family of three disjoint translates of disc that admits 3 geometric permutations, see Figure 1.1).

Chapter 3

Congruent Discs

In this chapter we deal with families of pairwise disjoint translates of a disc. A disc is a strictly convex and smooth set, hence by Theorem 2.1 there exists a minimal integer n such that any family of more than n disjoint congruent discs admits at most 2 geometric permutations (this result was obtained independently by Smorodinsky, Mitchell and Sharir [14, 15]). We also find the smallest such integer:

Theorem 3.1

1. *Any family of size ≥ 4 of disjoint congruent discs admits at most 2 geometric permutations.*
2. *All geometric permutations of a family of more than 3 disjoint congruent discs have representatives that coincide at all except possibly 2 consecutive places.*

The following lemmas play a central role in the proof of Theorem 3.1:

Lemma 3.2 *Let $\mathcal{A} = \{A, B, C, D\}$ be a family of disjoint congruent discs. If $< ABCD >$ is a geometric permutation for $\mathcal{A}$, then $< ACDB >$ can not be a geometric permutation for $\mathcal{A}$.*

Lemma 3.3 *Let $\mathcal{A} = \{A, B, C, D\}$ be a family of disjoint congruent discs. If $< ABCD >$ is a geometric permutation for $\mathcal{A}$, then $< CADB >$ can not be a geometric permutation for $\mathcal{A}$.*

In other words, the pair of geometric permutations $\{< ABCD >,$ $< ACDB >\}$ is incompatible for families of four disjoint congruent discs, and so is the pair $\{< ABCD >, < CADB >\}$.

Throughout this section it is assumed w.l.o.g. that all the congruent discs have radius 1. $P(a)$ shall denote the center of the disc A.

3.1 Two lemmas and an observation on T-families

Lemma 3.4 *Let $\mathcal{A} = \{A, B, C, D\}$ be a T-family. Let l_1 and l_2 be directed transversals of $\mathcal{A}$ intersecting at O, such that l_1 induces the permutation $(ACDB)$ and l_2 induces the permutation $(ABCD)$. Then one of five cases occurs:*

I. A crosses III, B crosses IV, C crosses III, D crosses I;

II. A crosses III, B crosses IV, C crosses II, D crosses I;

III. A crosses III, B crosses IV, C crosses III, D crosses II;

IV. A crosses III, B crosses IV, C crosses III, $O \in D$;

V. A crosses III, B crosses IV, $O \in C$, D crosses I.

(See Figure 3.1.)

Lemma 3.5 *Let $\mathcal{A} = \{A, B, C, D\}$ be a T-family. Let l_1 and l_2 be directed transversals of $\mathcal{A}$ intersecting at O, such that l_1 induces the permutation $(CADB)$ and l_2 induces the permutation $(ABCD)$. Then*
A crosses III, B crosses IV, C crosses II, D crosses I.

(See Figure 3.2.)

Note that Lemmas 3.4 and 3.5 deal with *representatives* of geometric permutations.

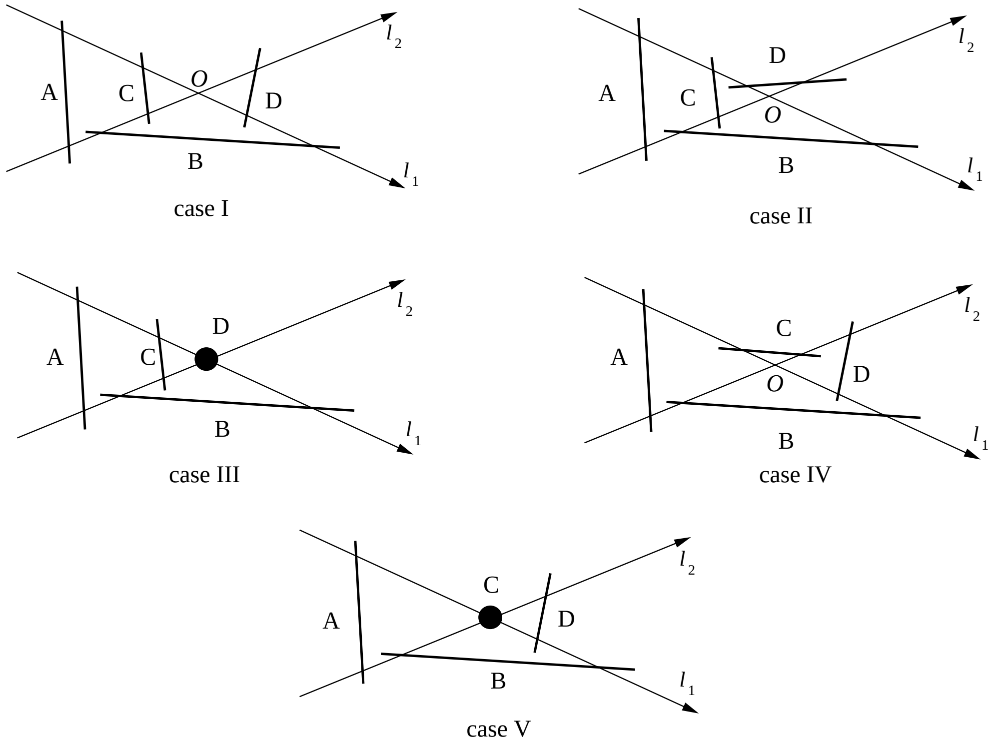


Figure 3.1: Possibilities for the pair $\{(ABCD), (ACDB)\}$.

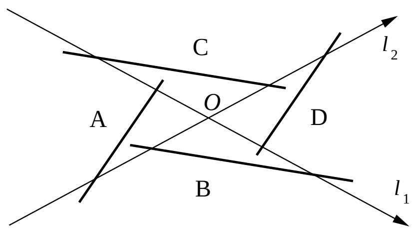


Figure 3.2: The only possibility for the pair $\{(ABCD), (CADB)\}$.

We use the following lemma proved by Katchalski, Lewis and Liu in [11]:

Lemma 3.6 *Let $\{A, B, C\}$ be a T-family, and l_1 and l_2 - directed transversals of this family, and let O be the point of intersection of l_1 and l_2. Then the following situation is impossible: A crosses quadrant I, B and C cross quadrant II such that $B \cap l_2$ and $C \cap l_2$ are between O and $A \cap l_2$.*

See Figure 3.3 for the illustration of Lemma 3.6.

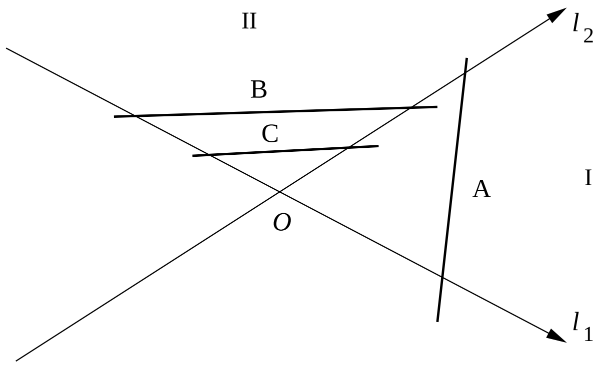


Figure 3.3: A situation which is impossible for translates by Lemma 3.6.

Proof of Lemma 3.4: Point O divides each of the permutations into two parts (One of them may be empty. It is also possible that O is contained in one of the sets). For two permutations there are 29 cases, which are presented in Figure 3.4. We observe that some cases contradict the convexity and the disjointness of the sets, and some cases contradict Lemma 3.6. In Figure 3.4 next to each case we list the sets that cause the contradiction (two sets if the case contradicts the convexity and the disjointness, three sets if the case contradicts Lemma 3.6). Only five cases remain, see Figure 3.1.
■

The **proof of Lemma 3.5** is analogous to that of Lemma 3.4. Only one case remains, see Figure 3.2.

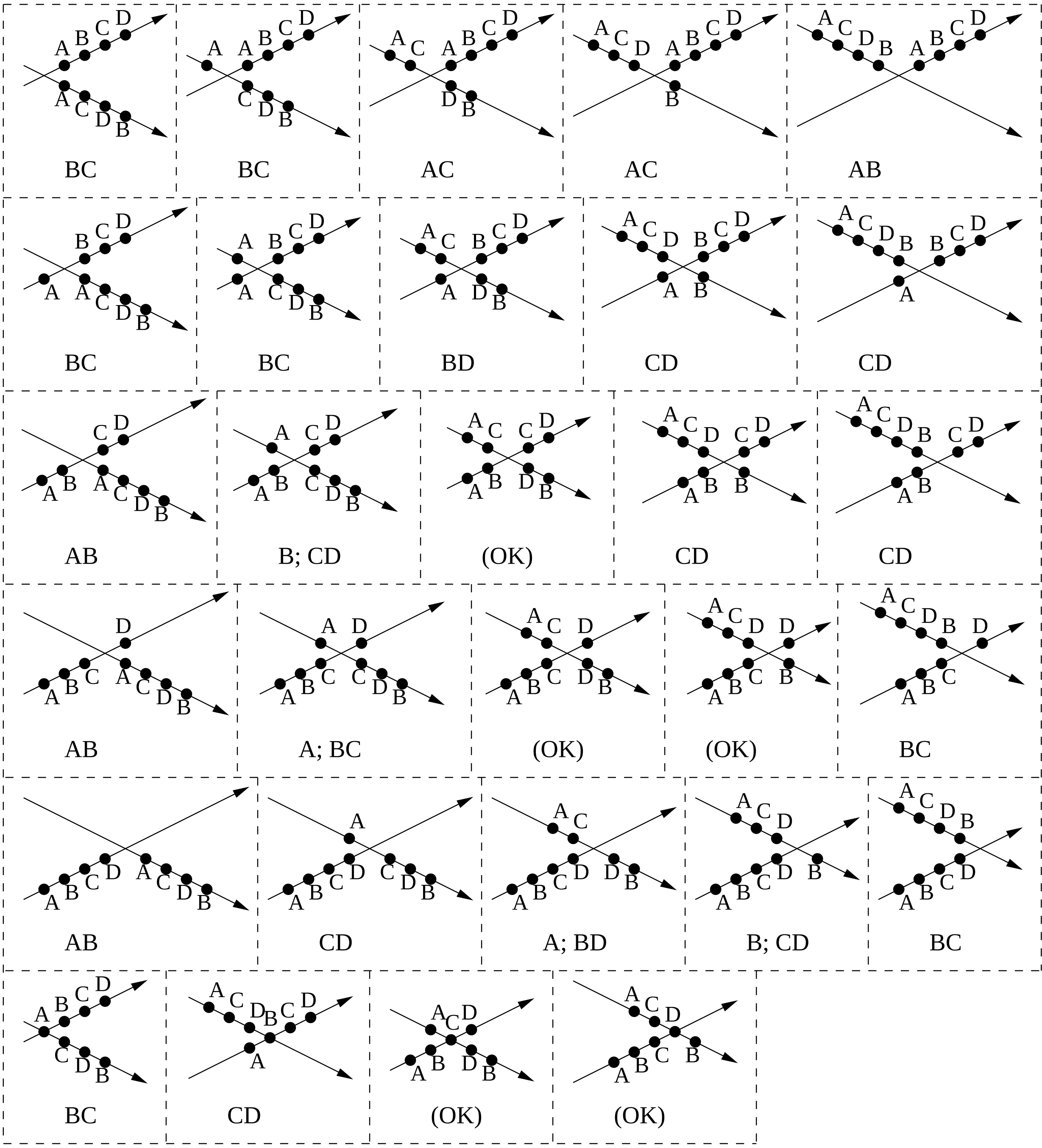


Figure 3.4: Illustration of the proof of Lemma 3.4.

3.2 Proof of Lemma 3.2

We begin by some technical lemmas.

Lemma 3.7 *Let $\angle MON$ be an angle, and let A, C be two disjoint congruent discs both crossing $\angle MON$. Then $\angle MON < 60°$.*

Proof: Assume w.l.o.g. that $OM \bigcap C$ is between O and $OM \bigcap A$ (then also $ON \bigcap C$ is between O and $ON \bigcap A$).

The discs A and C are disjoint, hence there exists a line k that crosses $\angle MON$ and separates A from C. The disc C crosses $\angle MON$, hence there exists a line l that crosses $\angle MON$ and separates C from point O. We can assume that k and l intersect at P forming an angle that contains C (see Figure 3.5). Translate C in the direction parallel to k such that C moves

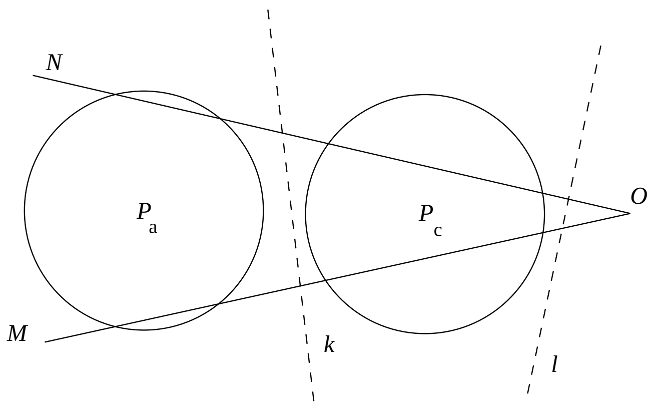


Figure 3.5: Discs A and C cross angle $\angle MON$.

away from l until it becomes tangent to one of the sides of $\angle MON$, say to OM. Translate A in the direction of OM until it becomes tangent to ON, and then translate A in the direction of ON until it becomes tangent to OM. A and C clearly remain disjoint and still cross the angle $\angle MON$.

Now C is tangent to OM, and A is tangent to both OM and ON. Denote $M_a = OM \bigcap A$, $M_c = OM \bigcap C$ (see Figure 3.6).

$P_aM_a = 1$, $OM_a > M_cM_a = P_cP_a > 2$; therefore $\tan(\angle MOP_a) < 1/2$, $\angle MOP_a < \arctan(1/2) \approx 26.57°$ and finally $\angle MON = 2\angle MOP_a < 60°$, proving the lemma.

■

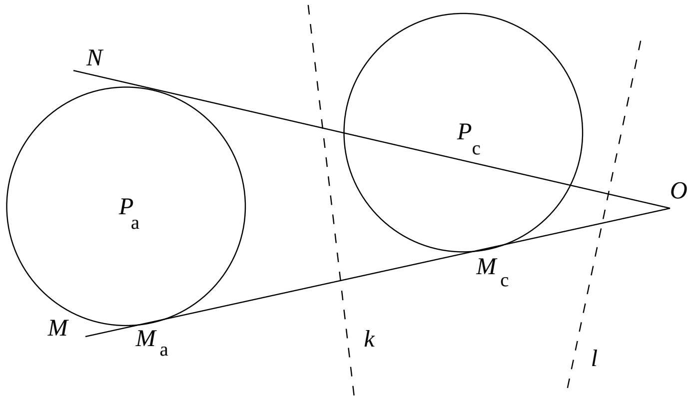


Figure 3.6: C is tangent to OM, A is tangent to both OM and ON.

Lemma 3.8 *Let l_1 and l_2 be two directed lines intersecting at O. Let A, C, D be disjoint congruent discs such that A crosses the quadrant III, D crosses the quadrant I, C either crosses one of the even quadrants or contains O. Then the angle between the positive directions of l_1 and l_2 is less than* $90°$.

Proof: Assume w.l.o.g. that C either crosses the quadrant II or contains O.

The discs A and C are disjoint, hence there exists a line k that crosses quadrant III and separates A from C. The discs C and D are disjoint, hence there exists a line l that crosses quadrant III and separates C from D (see Figure 3.7). It is possible to assume that k and l intersect. Translate A in the negative direction of l_1 until it becomes tangent to l_2, and then translate A in the negative direction of l_2 until it becomes tangent to l_1. Translate D in the positive direction of l_1 until it becomes tangent to l_2, and then translate D in the positive direction of l_2 until it becomes tangent to l_1. Translate C in the direction parallel to k such that C moves away from l until it becomes tangent to one of the lines l_1 or l_2 say to l_1 (see Figure 3.8).

Translate C in the negative direction of l_1 until it becomes tangent to l_2. We claim that still $l_1 \bigcap C$ is between $l_1 \bigcap A$ and O. Denote: $S = A \bigcap l_1$ and $T = D \bigcap l_2$, $V = C \bigcap l_1$, $W = C \bigcap l_2$ (see Figure 3.9). Now, since l still separates C from D, $|OV| = |OW| < |OT| = |OS|$. Now C is tangent to both l_1 and l_2.

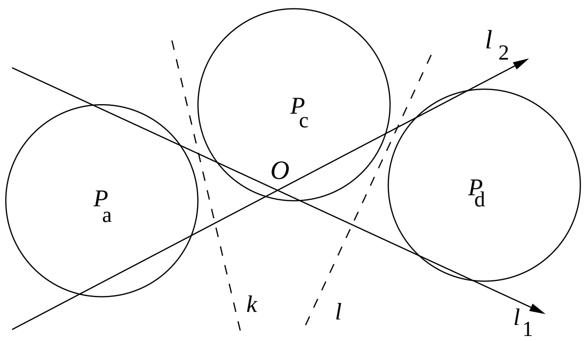


Figure 3.7: A crosses III, D crosses I, C either crosses II or contains O.

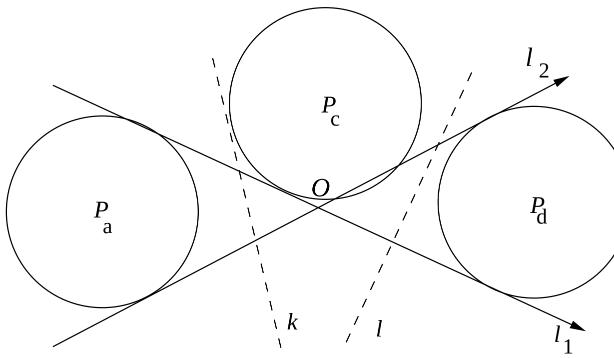


Figure 3.8: C is tangent to l_1.

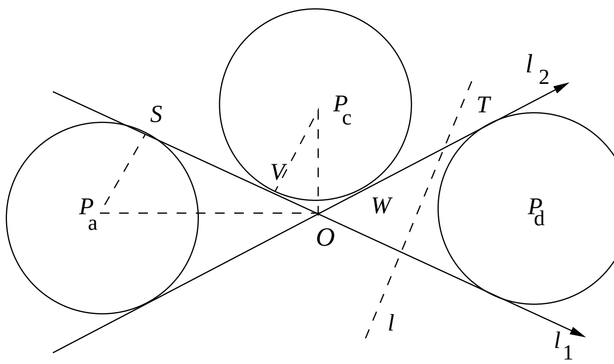


Figure 3.9: C is tangent to l_1 and l_2.

The angle between the positive directions of l_1 and l_2 is two times $\angle P_a OS$. $|OS| > |OV| \Rightarrow \angle P_a OS < \angle P_c OV$, but also $\angle P_a OS + \angle P_c OV = 90°$, therefore $\angle P_a OS < 45°$, proving the lemma.
■

Lemma 3.9 *Let l_1 and l_2 be two directed lines intersecting at O. Let B, C, D be disjoint congruent discs such that B crosses the quadrant IV, C crosses the quadrant III, D crosses the quadrant I, and $C \bigcap l_2$ is between $B \bigcap l_2$ and O, $D \bigcap l_1$ is between $B \bigcap l_1$ and O. Then the angle between the positive directions of l_1 and l_2 is greater than $60°$.*

Proof: Figure 3.10 shows the case described in the lemma.

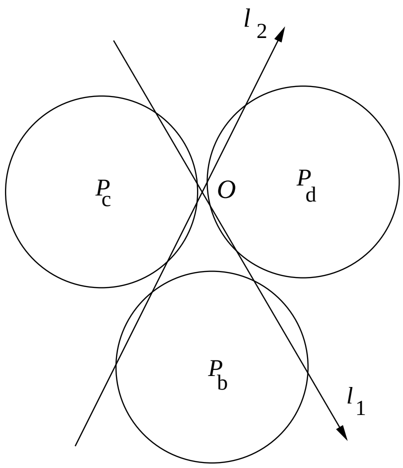


Figure 3.10: B crosses IV, C crosses III, D crosses I; $C \bigcap l_2$ is between $B \bigcap l_2$ and O, $D \bigcap l_1$ is between $B \bigcap l_1$ and O.

Translate B in the positive direction of l_1 until it becomes tangent to l_2, and then translate B in the negative direction of l_2 until it becomes tangent to l_1. Translate C in the negative direction of l_1 until it becomes tangent to l_2, and translate D in the positive direction of l_2 until it becomes tangent to l_1. The discs remain disjoint (see Figure 3.11).

Translate C in the negative direction of l_2 until it becomes tangent to B, and translate D in the positive direction of l_1 until it becomes tangent to B. Denote: $S = C \bigcap B$ and $T = D \bigcap B$ (see Figure 3.12).

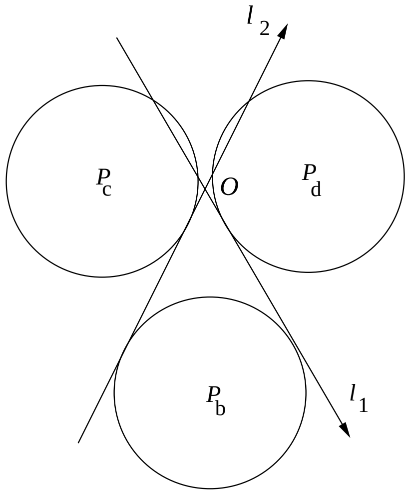


Figure 3.11: B is tangent to both l_1 and l_2, C is tangent to l_2, D is tangent to l_1.

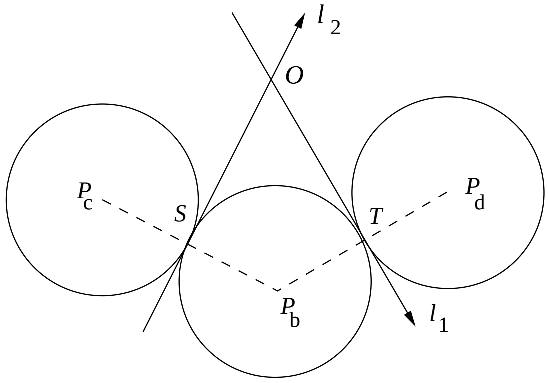


Figure 3.12: C and D are tangent to B.

Denote the angle between the positive directions of l_1 and l_2 by α. Then $\angle SOT = 90° - \alpha \Rightarrow \angle SP_bT = \alpha \Rightarrow \angle P_cP_bP_d = \alpha$. If $\alpha \leq 60°$, then $|P_cP_d| \leq 2$ contradicting the disjointness of the discs C and D. This contradiction proves the lemma.

■

We use Lemmas 3.7, 3.8, 3.9 to prove Lemma 3.2.

Proof of Lemma 3.2: Assume to the contrary that $\mathcal{A}$ admits both geometric permutations $< ABCD >$ and $< ACDB >$. Let l_1 and l_2 be transversals inducing geometric permutations $< ACDB >$ and $< ABCD >$ respectively. Turn l_1 and l_2 into directed lines so that l_1 and l_2 induce permutations $(ACDB)$ and $(ABCD)$ respectively. By Lemma 3.4, we have one of the five cases represented on Figure 3.13. It suffices to prove that none of these cases is possible when the sets are congruent discs. For each of the

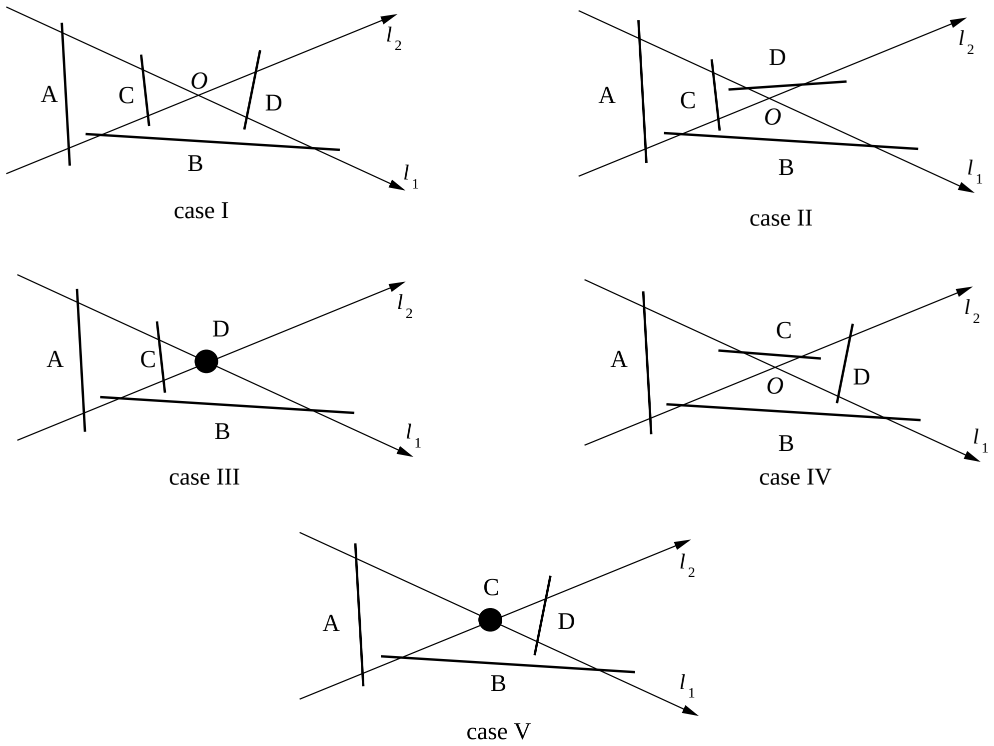


Figure 3.13: Possibilities for the pair $\{(ABCD), (ACDB)\}$.

cases, the angle between the positive directions of l_1 and l_2 is less than $90°$: for cases I, II, III it follows form Lemma 3.7, and for cases IV, V it follows from Lemma 3.8.

Assumptions and reductions. Translate A in the negative direction of l_1 until it becomes tangent to l_2. Then translate A in the negative direction of

l_2 until it becomes tangent to l_1. Then translate B in the positive direction of l_1 until it becomes tangent to l_2. Then translate B in the negative direction of l_2 until it becomes tangent to l_1. Now both A and B are tangent to both l_1 and l_2. Translate D in the positive direction of l_2 until it becomes tangent to l_1. If D now crosses the quadrant I or contains O, move D in the positive direction of l_1 such that after the translation the point $D \bigcap l_1$ is between O and $B \bigcap l_2$.

The discs remain disjoint and the lines l_1 and l_2 still induce the permutations $(ACDB)$, $(ABCD)$. (When we translate B in the negative direction of l_2, it is true because the angle between the positive directions of l_1 and l_2 is less than 90°.) Cases II and III reduce to case I.

Thus, we assume that both A and B are tangent to both l_1 and l_2, and that D crosses the quadrant I and is tangent to l_1. It remains to prove that cases I, IV, V are impossible.

Impossibility of case I. By Lemma 3.7, the angle between the positive directions of l_1 and l_2 is *less* than 60°. On the other hand, by Lemma 3.9, the angle between the positive directions of l_1 and l_2 is *greater* than 60°. This contradiction proves the impossibility of case I.

Impossibility of cases IV and V. Translate D in the positive direction of l_1 until it becomes tangent to B. Since the angle between the positive directions of l_1 and l_2 is less than 90°, l_2 still intersects D.

If C is not tangent to l_2, rotate l_2 in such a way that it remains tangent to B and the point of tangency moves clockwise along the boundary of B, until l_2 becomes tangent to C (it obviously happens *before* l_2 misses A or D), and then translate A in the negative direction of l_1 until it becomes tangent to l_2.

Then translate C in the negative direction of l_2 until it becomes tangent to A (it happens *before* C contains O, since otherwise we can translate D a little in the negative direction of l_2 and obtain case I which is impossible).

Now we have the following situation: A, B, D are tangent to l_1, A, B, C are tangent to l_2; B is tangent to D, A is tangent to C (see Figure 3.14. In this figure the discs C and D intersect. In fact, we shall prove that they *must* intersect; the impossibility of this case will follow from this); all other pairs of discs are disjoint.

Let $M_a = A \bigcap l_2$, $M_c = C \bigcap l_2$, $N_a = A \bigcap l_1$ and $N_b = B \bigcap l_1 \; = D \bigcap l_1$.

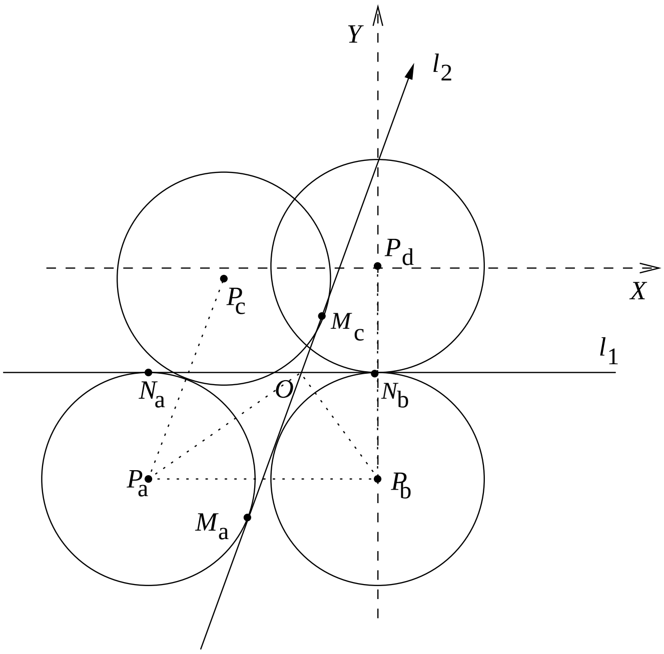


Figure 3.14: A, B, D are tangent to l_1; A, B, C are tangent to l_2; B is tangent to D; A is tangent to C.

Let α be the angle $\angle P_a O M_A$. This angle is half of the angle between the positive directions of l_1 and l_2, thus $\alpha < 45°$ and $\tan\alpha < 1$. Since $|M_a O| \le |M_a M_c| = |P_a P_c| = 2$, $\tan\alpha \ge 1/2$.

Consider a coordinate system with the origin at P_d and the X-axis parallel to l_1. In this system:

- $P_D = (0, 0)$;
- $P_B = (0, -2)$;
- $P_A = (-\tan\alpha - \frac{1}{\tan\alpha}, -2)$:
  $\angle N_a O P_a = \alpha, \angle N_b O P_b = 90° - \alpha$, hence $|N_a O| = \frac{1}{\tan\alpha}$ and $|M_b O| = \tan\alpha$, and finally $|N_a N_b| = \tan\alpha + \frac{1}{\tan\alpha}$.
- $P_C = (-\tan\alpha - \frac{1}{\tan\alpha} + 2\cos 2\alpha, -2 + 2\sin 2\alpha)$:
  $|P_a P_c| = 2$, $\angle P_c P_a P_b = 2\alpha$, thus, the projection of $P_a P_c$ to the X-axis has length $2\cos 2\alpha$, and the projection of $P_a P_c$ to the axis Y-axis has

length $2\sin 2\alpha$.

Let $c = \cot\alpha$. We saw that $1/2 \le \tan\alpha < 1$, hence $1 < c \le 2$. Using:

$$\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-1/c^2}{1+1/c^2} = \frac{c^2-1}{c^2+1},$$
$$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2/c}{1+1/c^2} = \frac{2c}{c^2+1},$$

we obtain:
$P_d = (0,0)$, $P_c = (-\frac{1}{c} - c + \frac{2(c^2-1)}{c^2+1}, -2 + \frac{4c}{c^2+1})$.

We shall prove that $|P_cP_d| < 2$, and this will contradict the disjointness of the discs C and D.

We have to prove that

$$(-\tfrac{1}{c} - c + \tfrac{2(c^2-1)}{c^2+1})^2 + (-2 + \tfrac{4c}{c^2+1})^2 < 2.$$

Now:

$$-\frac{1}{c} - c + \frac{2(c^2-1)}{c^2+1} =$$

$$= \frac{-(c^2+1)-c^2(c^2+1)+c(2c^2-2)}{c(c^2+1)} =$$

$$= \frac{-c^2-1-c^4-c^2+2c^3-2c}{c(c^2+1)} =$$

$$= -\frac{c^4-2c^3+2c^2+2c+1}{c(c^2+1)};$$

and

$$-2 + \frac{4c}{c^2+1} =$$

$$= \frac{-2(c^2+1)+4c}{c^2+1} =$$

$$= \frac{-2c^2+4c-2}{c^2+1} =$$

$$= -\frac{2c^3-4c^2+2c}{c(c^2+1)},$$

thus,

$$|P_cP_d|^2 = (\frac{c^4-2c^3+2c^2+2c+1}{c(c^2+1)})^2 + (\frac{2c^3-4c^2+2c}{c(c^2+1)})^2 =$$

$$= \frac{c^8+4c^6+4c^4+4c^2+1-4c^7+4c^6+4c^5+2c^4-8c^5-8c^4-4c^3+8c^3+4c^2+4c}{c^2(c^4+2c^2+1)} +$$

$$+ \frac{4c^6+16c^4+4c^2-16c^5+8c^4-16c^3}{c^2(c^4+2c^2+1)} =$$

$$= \frac{c^8-4c^7+12c^6-20c^5+22c^4-12c^3+12c^2+4c+1}{c^6+2c^4+c^2}.$$

It suffices to prove that

$$c^8 - 4c^7 + 12c^6 - 20c^5 + 22c^4 - 12c^3 + 12c^2 + 4c + 1 < 4(c^6 + 2c^4 + c^2),$$

or that

$$c^8 - 4c^7 + 8c^6 - 20c^5 + 14c^4 - 12c^3 + 8c^2 + 4c + 1 < 0.$$

Recall that $1 < c \leq 2$, therefore $c - 1 > 0$ and $c - 2 \leq 0$. Hence

$$c^7 - 3c^6 + 5c^5 - 15c^4 - c^3 - 13c^2 - 5c - 1 =$$

$$= c^7 - c^6 - 2c^6 + 5c^5 - 10c^4 - 5c^4 - c^3 - 13c^2 - 5c - 1 =$$

$$= \underbrace{c^6(c-2)}_{\leq 0} + \underbrace{5c^4(c-2)}_{\leq 0} + \underbrace{(-c^6 - 5c^4 - c^3 - 13c^2 - 5c - 1)}_{<0} < 0;$$

and, finally,

$$c^8 - 4c^7 + 8c^6 - 20c^5 + 14c^4 - 12c^3 + 8c^2 + 4c + 1 =$$

$$= \underbrace{(c-1)}_{>0}\underbrace{(c^7 - 3c^6 + 5c^5 - 15c^4 - c^3 - 13c^2 - 5c - 1)}_{<0} < 0,$$

proving that $|P_cP_d| < 2$.

This contradicts the disjointness of the discs C and D and thus proves the impossibility of cases IV and V.

■

3.3 Proof of Lemma 3.3

We begin with the following observation:

Observation 3.10 *Let S be a disc, let O be a point, and let $\overrightarrow{OX}$, $\overrightarrow{OY}$ be two rays such that $\overrightarrow{OX}$ is tangent to S, and $\overrightarrow{OY}$ intersects S. Let $M = OX \bigcap S$ and let N be the point of $OY \bigcap S$ closest to O. Then $|ON| \leq |OM|$.*

Proof: Denote by K the point of $OY \bigcap S$ farthest from O. By the "tangent and secant" theorem, $|OM|^2 = |ON||OK|$ (See Figure 3.15). The observation follows immediately.

■

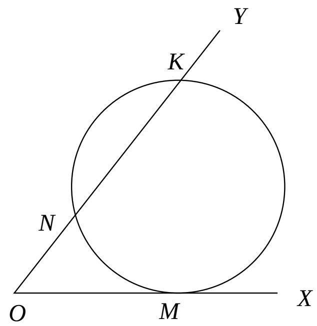


Figure 3.15: $|ON| \leq |OM|$.

Proof of Lemma 3.3: Assume to the contrary that $\mathcal{A}$ admits both geometric permutations $< ABCD >$ and $< CADB >$. Let l_1 and l_2 be transversals inducing geometric permutations $< CADB >$ and $< ABCD >$ respectively. Turn l_1 and l_2 into directed lines so that they induce permutations $(CADB)$ and $(ABCD)$ respectively. By Lemma 3.5, we have the case represented on Figure 3.16. It suffices to prove that this case is impossible with congruent discs.

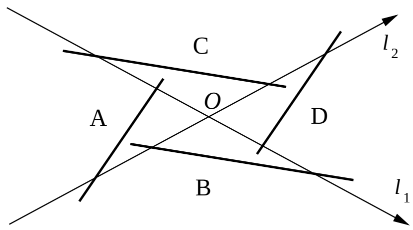


Figure 3.16: The only possibility for the pair $\{(ABCD), (ACDB)\}$.

Translate A in the negative direction of l_2 until it becomes tangent to l_1. Then translate B in the positive direction of l_1 until it becomes tangent to l_2. Then translate C in the negative direction of l_1 until it becomes tangent to l_2. Then translate D in the positive direction of l_2 until it becomes tangent to l_1. The discs remain disjoint, and the lines l_1 and l_2 still induce permutations $(CADB), (ABCD)$. Now A and D are tangent to l_1, and B and C are tangent to l_2.

For $i = 1, 2$, denote the point of $A \bigcap l_i$ closest to O by A_i. In a similar way define B_i, C_i and D_i. Using Lemma 3.5 (see Figure 3.16) and Observation

3.10, we have

$$|OA_2| \leq |OA_1| < |OC_1| \leq |OC_2| < |OD_2| \leq |OD_1| < |OB_1| \leq |OB_2| < |OA_2|$$
$$\Downarrow$$
$$|OA_2| < |OA_2|.$$

This contradiction proves the lemma.
■

3.4 Proof of Theorem 3.1

Proof: Let $\mathcal{A}$ be a family of disjoint congruent discs. First we prove that if $|\mathcal{A}| = 4$ then $\mathcal{A}$ admits at most 2 geometric permutations.

Let $\mathcal{A} = \{A, B, C, D\}$. By Lemmas 2.10 and 3.2 it is impossible that $\mathcal{A}$ admits two geometric permutations from the same column in the following table:

$< ABCD >$	$< ACDB >$
$< ADBC >$	$< ABDC >$
$< ACDB >$	$< ADCB >$
$< BCAD >$	$< BACD >$
$< BADC >$	$< BDAC >$
$< CABD >$	$< CBAD >$

It follows that $\mathcal{A}$ can not admit more than 2 geometric permutations.

Now we extend this for the case $|\mathcal{A}| > 4$.

By Theorem 2.12, if $\mathcal{A}$ admits 3 geometric permutations, then they are of the form $\{< WK_1W' >, < WK_2W' >, < WK_3W' >\}$ where K_1, K_2, K_3 are representatives of different geometric permutations of four sets. But we have just proved that three different geometric permutations can not coexist on four members of $\mathcal{A}$. Hence $\mathcal{A}$ admits at most 2 geometric permutations.

Now assume that $\mathcal{A}$ admits two geometric permutations. It follows from Theorem 1.1, that these geometric permutations are of the form $< WK_1W' >$, $< WK_2W' >$ where K_1, K_2 are representations of different geometric permutations of four sets. Let A, B, C, D be the discs on which these geometric permutations disagree. The family $\{A, B, C, D\}$ admits two geometric permutations. There are 12 geometric permutations on four sets (they are listed in Lemma 2.11), and there are six types of pairs of these geometric permutations (up to relabeling):

1. $\{< ABCD >, < BADC >\}$;
2. $\{< ABCD >, < CBAD >\}$;
3. $\{< ABCD >, < ACDB >\}$;
4. $\{< ABCD >, < CADB >\}$;
5. $\{< ABCD >, < BACD >\}$;
6. $\{< ABCD >, < ACBD >\}$.

The first and the second types are impossible for T-families by Lemma 2.10. The third and the fourth types are impossible for families of disjoint congruent discs by Lemmas 3.2 and 3.3. Thus, if $\mathcal{A}$ is of size 4, the only possible pairs of geometric permutations are $\{< ABCD >, < BACD >\}$ and $\{< ABCD >, < ACBD >\}$. If $\mathcal{A}$ is of size 5 or more, then the pair of geometric permutations is $\{< WK_1W' >, < WK_2W' >\}$ where, w.l.o.g., $K_1 = (ABCD)$, and K_2 is one of the following: $(BACD)$, $(DCAB)$, $(ACBD)$, $(DBCA)$. But the pairs $\{< WABCDW' >, < WDCABW' >\}$ and $\{< WABCDW' >, < WDBACW' >\}$ are incompatible by Lemma 2.10, since at least one of W, W' is not empty. Thus, the only pairs of geometric permutations for families of disjoint congruent discs are $\{< WABCDW' >, < WBACDW' >\}$ and $\{< WABCDW' >, < WACDBW' >\}$. Each of them can be written as $\{< WABW' >, < WBAW' >\}$. This proves the second part of the theorem.
■

Chapter 4

Possible Triples of Geometric Permutations for T-families

Theorem 1.1 asserts that a T-family can not have more than 3 geometric permutations. In this section we find all possible triples of geometric permutations for T-families:

Theorem 4.1 *Let $\mathcal{A}$ be a T-family of size ≥ 4 that admits 3 geometric permutations. Then the triple of geometric permutations is one of the following:*

1. $\{< W123W' >, < W213W' >, < W132W' >\}$,
2. $\{< W1234W' >, < W2134W' >, < W2413W' >\}$.

Theorem 4.1 refines Theorem 2.12 by Tverberg [17] and corrects one of the statements from [17].

First we prove this for T-families of size 4 and then generalize the result to T-families of size ≥ 4.

4.1 Proof of Theorem 4.1 for $|\mathcal{A}| = 4$

Proof: Let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ be a T-family with 3 geometric permutations. By Observation 2.13, there are just four triples of geometric permutations that do not contradict Lemma 2.11:

1. $\{< 1234 >, < 1342 >, < 1423 >\}$;

2. $\{< 1234 >, < 1324 >, < 1243 >\}$;

3. $\{< 1234 >, < 4132 >, < 2431 >\}$;

4. $\{< 1234 >, < 2134 >, < 2413 >\}$.

We prove that the first and the third triples are impossible for T-families and present examples of T-families that admit the second and the fourth triples.

The triple $\{< 1234 >, < 1342 >, < 1423 >\}$ is impossible for T-families of size 4: Assume to the contrary that $\mathcal{A}$ admits geometric permutations $< 1234 >$, $< 1342 >$, $< 1423 >$. Let l_1, l_2 and l_3 be directed transversals of $\mathcal{A}$ inducing permutations (1234), (1342), (1423) respectively. Choose a point O in the plane, and translate the transversals so that they will contain O. Because of the cyclic symmetry of the permutations, we obtain w.l.o.g. one of the two cases shown in Figure 4.1:

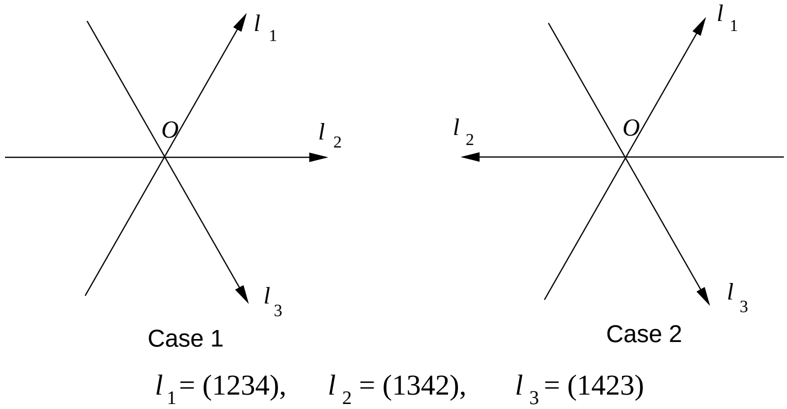


Figure 4.1: Two cases: l_1, l_2 and l_3 translated to O.

Case 1:

Let $-l_2$ be the line obtained from l_2 by reversing its direction. There is no line m in the plane such that l_1, -l_2 and l_3 meet two halfplanes bounded by m in the same order. On the other hand, the lines l_1, -l_2 and l_3 meet A_2 before A_3. Thus, choosing m to be a line separating A_2 from A_3, we obtain a contradiction.

Case 2:

There is no line m in the plane such that l_1, l_2 and l_3 meet two halfplanes bounded by m in the same order. On the other hand, the lines l_1, l_2 and l_3 meet A_1 before A_2. Thus, choosing m to be a line separating A_1 from A_2, we obtain a contradiction.

Note that in fact we proved that the triple $\{< 1234 >, < 1342 >, < 1423 >\}$ is impossible for families of four pairwise disjoint arbitrary convex sets (that are not necessarily translates).

The triple $\{< 1234 >, < 4132 >, < 2431 >\}$ is impossible for T-families of size 4: Assume to the contrary that $\mathcal{A}$ admits geometric permutations $< 1234 >$, $< 4132 >$, $< 2431 >$. Let l_1, l_2 and l_3 be directed transversals of $\mathcal{A}$ inducing permutations (1234), (4132), (2431) respectively. Choose a point O in the plane, and translate the transversals so that they will contain O. Because of the cyclic symmetry of the permutations, we obtain w.l.o.g. one of the two cases shown in Figure 4.2:

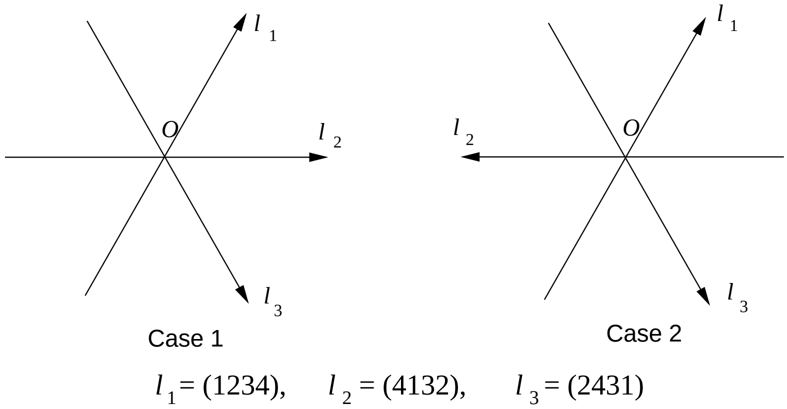


Figure 4.2: Two cases: l_1, l_2 and l_3 translated to O.

Case 1:

Let $-l_2$ be the line obtained from l_2 by reversing its direction. There is no line m in the plane such that l_1, -l_2 and l_3 meet two halfplanes bounded by m in the same order. On the other hand, the lines l_1, -l_2 and l_3 meet A_2 before A_4. Thus, choosing m to be a line separating A_2 from A_4, we obtain a contradiction.

Case 2:

Tverberg proved [16, 17] that if the T-family $\{c_i + K\}$ admits certain geometric permutations, then the family $\{c_i + \frac{K-K}{2}\}$ is also T-family and it admits exactely the same geometric permutations. Therefore it is possible to assume that A_1, A_2, A_3 and A_4 are translates of a *centrally symmetric* convex set. By standard arguments it is possible to assume that they are *strictly convex.*

Construct a hexagon by taking six points on the boundary of A_1: in both halfplanes bounded by l_1 (l_2, l_3) take the points on the boundary of A_1 farthest from l_1 (l_2, l_3). H_1, the convex hull of these points is a centrally symmetric convex hexagon inscribed in A_1. In a similar way construct hexagons H_2, H_3 and H_4 inscribed in A_2, A_3, A_4 respectively. H_2, H_3 and H_4 are clearly translates of H_1. Let $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$. $\mathcal{H}$ is a family of pairwise disjoint translates of a convex hexagon H, and it admits the same geometric permutations as $\mathcal{A}$. Therefore **it suffices to prove that Case 2 is impossible for T-families whose members are translates of a centrally symmetric convex hexagon.** Note that it follows from Observation 2.4 that the hexagons are not degenerate: for each pair of opposite sextants (see Figure 4.2, Case 2), H has an edge e which is parallel to a line m such that m contains O, belongs to this pair of opposite sextants and is different from the lines that form these sextants.

One of the edges of H_3 separates H_1 from H_3. Consider a line m_1 that contains an edge e_1 of H_3 and separates H_3 from H_1. Translate m_1 so that it will contain O. Case 2 splits into six subcases (Figure 4.3).

Case 2.1 is impossible because l_2 meets H_1 before H_3,
case 2.2 is impossible because l_1 meets H_1 before H_3,
case 2.3 is impossible because l_1 meets H_1 before H_3,
case 2.5 is impossible because l_1 meets H_1 before H_3,
case 2.6 is impossible because l_2 meets H_1 before H_3.

Thus, only case 2.4 is possible: m_1 belongs to the pair of opposite sextants formed by l_1 and l_2. Moreover, it follows that e_1 is the *only* edge of H_3 such that a line through it separates H_1 from H_3.

Consider a line m_2 that contains an edge e_2 of H_3 and separates H_3 from

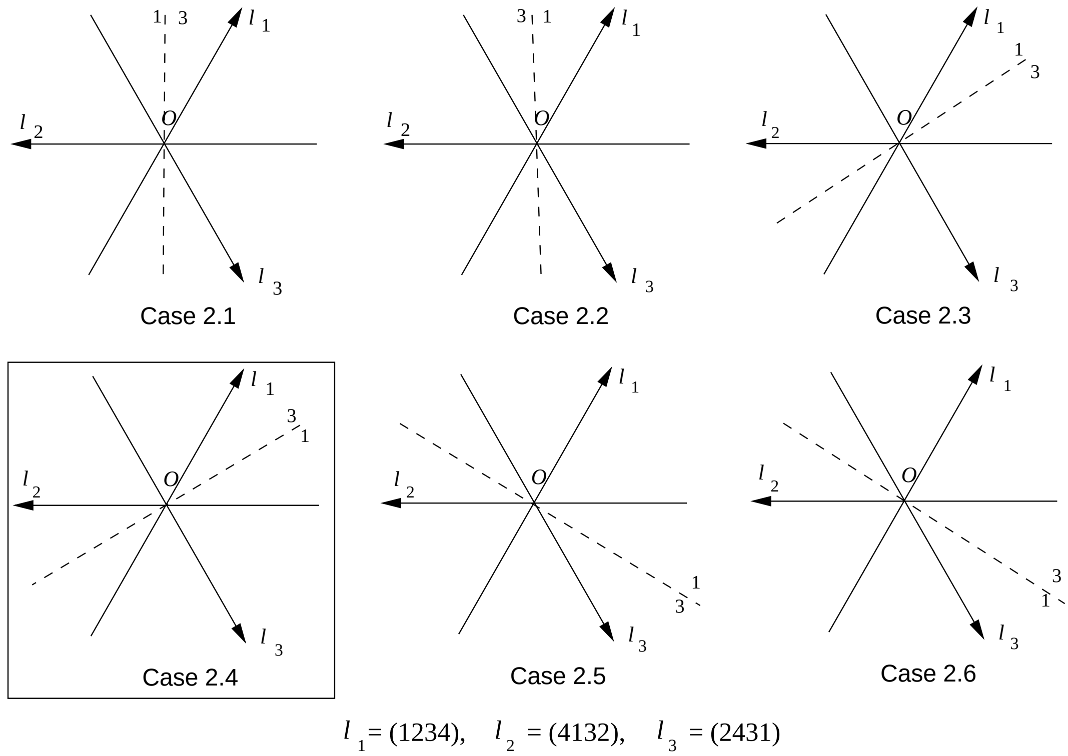
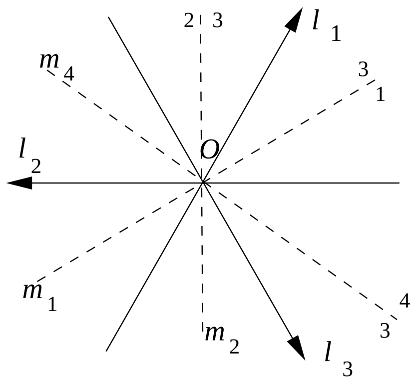


Figure 4.3: Six subcases of Case 2 from Figure 4.2.

H_2 and consider a line m_4 that contains an edge e_4 of H_3 and separates H_3 from H_4. Translate m_2 and m_4 so that they will contain O. Analyzing six cases, as was done above, we conclude that the directions of m_2 and m_4 are as shown in Figure 4.4: m_2 belongs to the pair of opposite sextants formed by l_1 and l_3, m_4 belongs to the pair of opposite sextants formed by l_2 and l_3. We also conclude that e_2 and e_4 are the *only* edges of H_3 that separate H_3 form H_2 and from H_4 respectively. Say that the edges e_1, e_2 and e_4 are *separators*. Note that the edges of H_3 that are separators alternate with the edges that are not separators. Let $\tilde{e_1}$, $\tilde{e_2}$, $\tilde{e_4}$ be the edges of H_3 opposite to e_1, e_2, e_4 respectively.



Case 2.4

$l_1 = (1234)$, $l_2 = (4132)$, $l_3 = (2431)$

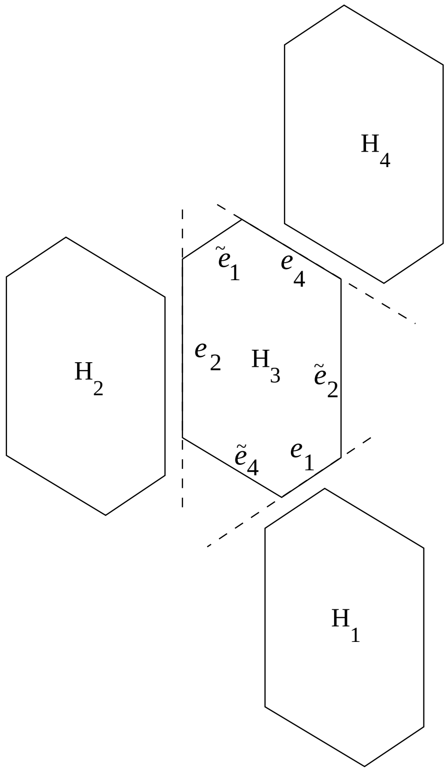


Figure 4.4: For $i = 1, 2, 4$, e_i is contained in a line that separates H_i from H_3.

Assume, after using an affine transformation of the plane if needed, that all the angles of H are 120°. Assume w.l.o.g. that $|e_1| \le |e_2|$ and $|e_1| \le |e_4|$. Let $e_2 = AB$, $\tilde{e_4} = BC$, $e_1 = CD$, let E be the point of intersection of the lines containing e_2 and e_1 and let XY be the edge of H_1 corresponding to $\tilde{e_1}$ (see Figure 4.5).

Since the line BC does not separate H_3 from H_1, the line BC separates E from Y. Since $|CD| = |XY| \le |CE| = |CB|$, the line AB does not intersect H_1. It follows that the line AB separates H_2 from H_1. Thus, the line AB separates H_2 from H_3 and also separates H_2 from H_1. This contradicts the assumption that the family $\mathcal{H}$ admits the geometric permutation $< 1234 >$.

Remark If e_2 is the shortest edge, then $\mathcal{H}$ can not admit the geometric permutation $< 2431 >$ and if e_4 is the shortest edge, then $\mathcal{H}$ can not admit the geometric permutation $< 4132 >$.

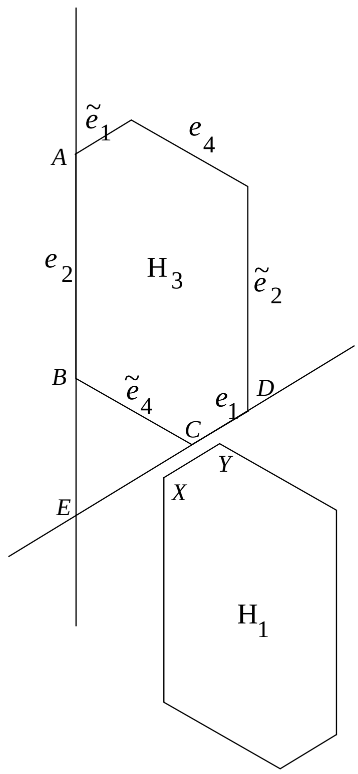


Figure 4.5: Triple $\{< 1234 >, < 4132 >, < 2431 >\}$ is impossible for T-families.

Triple $< 1234 >, < 1324 >, < 1243 >$ **is possible for** T**-families of size 4:** see Figure 4.6. Another example can be found in [11].

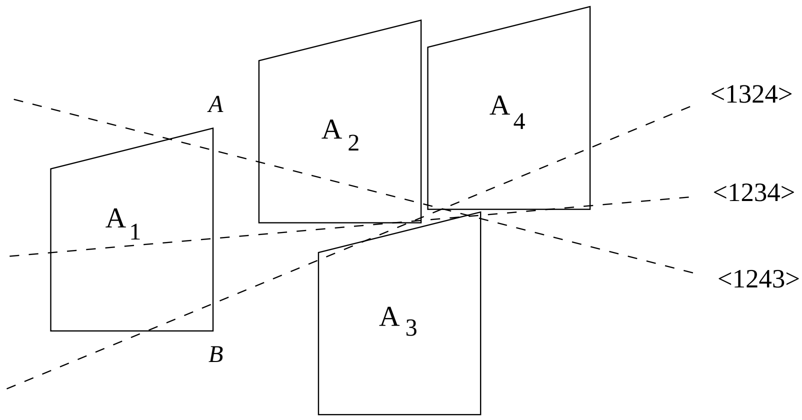


Figure 4.6: A T-family with the triple $\{< 1234 >, < 1324 >, < 1243 >\}$

Triple $< 1234 >, < 2134 >, < 2413 >$ **is possible for** T**-families of size 4:** Figures 4.7 and 4.8 show the following family:

$S1 = conv\{(-24, 0), (-18, 1), (0, 48), (-18, 264), (-24, 263), (-42, 216)\}$

$S2 = S1 + (0, -264) =$
$conv\{(-24, -264), (-18, -263), (0, -216), (-18, 0), (-24, -1), (-42, -48)\}$

$S3 = S1 + (30, 0) =$
$conv\{(6, 0), (12, 1,), (30, 48), (12, 264), (6, 263), (-12, 216)\}$

$S4 = S1 + (42, -249) =$
$conv\{(18, -249), (24, -248), (42, -201), (24, 15), (18, 14), (0, -33)\}$.

The translates are disjoint, and
the X-axis is a transversal with geometric permutation $< 1234 >$,
the Y-axis is a transversal with geometric permutation $< 3142 >$,
the line $Y = X/3 + 7$ is a transversal with geometric permutation $< 2134 >$.

This example corrects a statement from [17] that claimed that this triple is impossible for T-families.

■

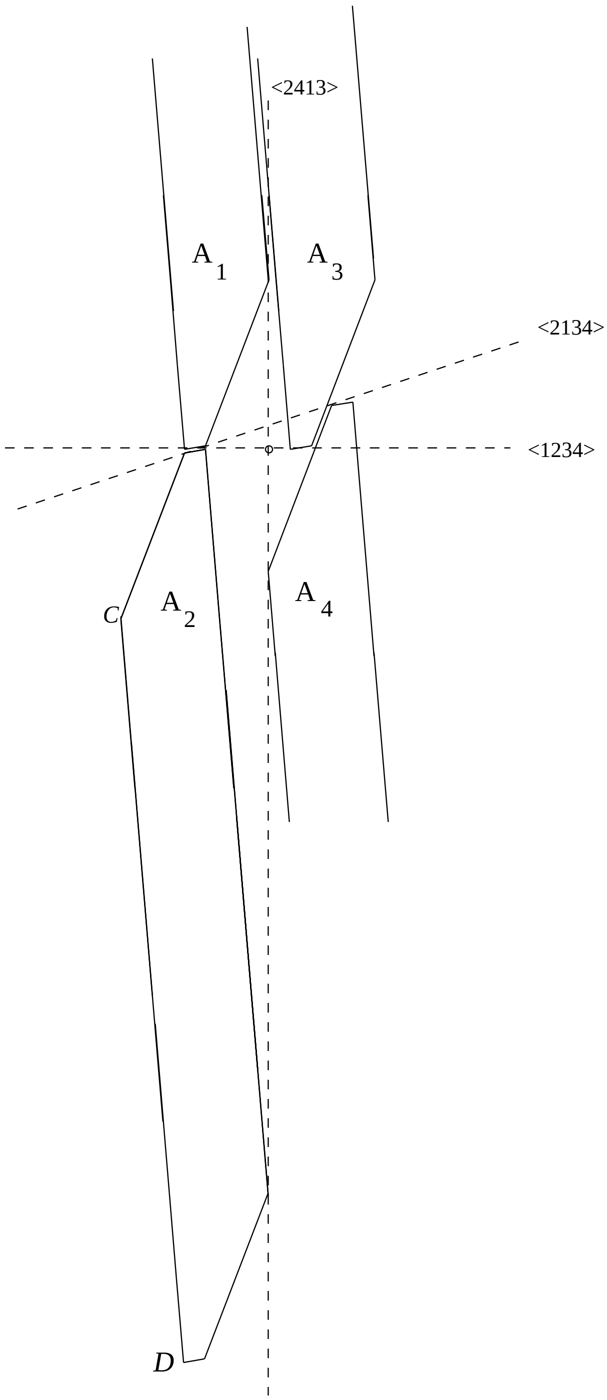


Figure 4.7: A T-family with the triple $\{< 1234 >, < 2134 >, < 2413 >\}$.

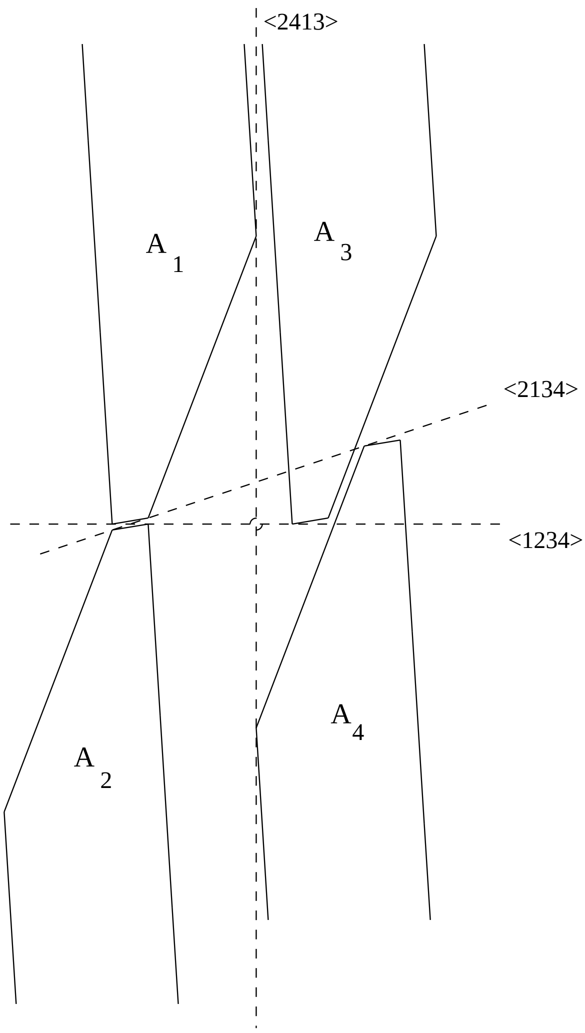


Figure 4.8: A T-family with the triple $\{< 1234 >, < 2134 >, < 2413 >\}$ (Figure 4.7 magnified).

4.2 Proof of Theorem 4.1 for $|\mathcal{A}| > 4$

Proof: Let $\mathcal{A}$ be a T-family admitting 3 geometric permutations. By Theorem 2.12 (proved by Tverberg [17]), the triple of geometric permutations is of the form $\{< WK_1W' >, < WK_2W' >, < WK_3W' >\}$ where K_1, K_2, K_3 are representatives of distinct geometric permutations of four sets. We proved in Section 4.1 that there are just two possible triples for T-families of size 4: $\{< 1234 >, < 1324 >, < 1243 >\}$ and $\{< 1234 >, < 2134 >, < 2413 >\}$. Thus the geometric permutations of $\mathcal{A}$ form one of the following triples:

1. $\{< W1234W' >, < W1324W' >, < W1243W' >\}$,
2. $\{< W1234W' >, < W1324W' >, < W3421W' >\}$,
3. $\{< W1234W' >, < W4231W' >, < W1243W' >\}$,
4. $\{< W1234W' >, < W4231W' >, < W3421W' >\}$,
5. $\{< W1234W' >, < W2134W' >, < W2413W' >\}$,
6. $\{< W1234W' >, < W2134W' >, < W3142W' >\}$,
7. $\{< W1234W' >, < W4312W' >, < W2413W' >\}$,
8. $\{< W1234W' >, < W4312W' >, < W3142W' >\}$.

Since at least one of W or W' is not empty, among these triples all but the first and the fifth contradict Lemma 2.10 that asserts that geometric permutations $< abcd >$ and $< cbad >$ can not coexist. Thus, $\{< W1234W' >, < W1324W' >, < W1243W' >\}$ and $\{< W1234W' >, < W2134W' >, < W2413W' >\}$ are the only possibilities. For each $n > 4$ it is possible to construct T-families that admit triples of geometric permutations of these types. In order to construct such families one has to make the edges parallel to AB in the set from Figure 4.6 (or the edges parallel to CD in the set from Figure 4.8) longer, and it will then be possible to add sets to the families shown in these Figures, obtaining thus T-families of size n that admit triples $\{< W1234W' >, < W1324W' >, < W1243W' >\}$ (or $\{< W1234W' >, < W2134W' >, < W2413W' >\}$).
■

Chapter 5

Open Problems

Here we discuss some open problems related to geometric permutations.

Recall that a family $\mathcal{A}$ has *property* $\mathbf{T}_r$ if each subfamily of $\mathcal{A}$ of size $\leq r$ has a transversal and that $\mathcal{A}$ has *property* $\mathbf{T}$ if the entire family has a transversal. Another related property is the following: $\mathcal{A}$ has *property* $\mathbf{T}-k$ if a subfamily of $\mathcal{A}$ of size $|\mathcal{A}|-k$ has a transversal. There are several results and conjectures on these properties for T-families:

1. For T-families $\mathbf{T}_5 \Rightarrow \mathbf{T}$ (Tverberg 1989 [16]).

2. For T-families of squares $\mathbf{T}_4 \Rightarrow \mathbf{T}-2$ (Katchalski and Lewis 1982 [9]).

3. There exists an integer k such that for T-families $\mathbf{T}_3 \Rightarrow \mathbf{T}-k$ (Katchalski and Lewis 1980 [8]).

Katchalski and Lewis proved that in the last theorem $k \leq 603$. Tverberg improved this to $k \leq 108$ [17], but it was conjectured (this is the Katchalski-Lewis conjecture) that $k = 2$. This has not been settled even for the cases of discs and squares. It is believed (Tverberg [17]) that the study of geometric permutations could be useful for this problem.

Another related problem is that of estimating the maximal possible number of geometric permutations for families of n disjoint convex sets in $\mathbb{R}^d$. It is only known that this number is $\Omega(n^{d-1})$ (Katchalski, Lewis and Zaks [13]) and $O(n^{2d-2})$ (Wenger [19]). It is also known that for families of n disjoint balls in $\mathbb{R}^d$ this number is $\Theta(n^{d-1})$ (Smorodinsky, Mitchell and Sharir [14, 15]).

It is not known whether there is a constant bound on the number of geometric permutations for families of n disjoint congruent balls in $\mathbb{R}^3$.

Unlike the situation in the plane it is still not known whether there exists an integer r such that $\mathbf{T}_r \Rightarrow \mathbf{T}$ for all families of disjoint translates of a convex set in $\mathbb{R}^3$.

It is also unknown what pairs of permutations on $\{1, 2, \dots, n\}$ can be realized as geometric permutations of families of n disjoint translates of a convex set in $\mathbb{R}^3$ (it is easy to see that any two permutations on $\{1, 2, \dots, n\}$ can be realized as a pair of geometric permutations for a family of n disjoint convex sets in $\mathbb{R}^3$).

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