

# Geometric Permutations of Large Families of Translates

*A. Asinowski*

*A. Holmsen*

*M. Katchalski*

*H. Tverberg*

## Abstract

Let  $F$  be a finite family of disjoint translates of a compact convex set  $K$  in  $\mathbf{R}^2$ , and let  $l$  be an ordered line meeting each of the sets. Then  $l$  induces in the obvious way a total order on  $F$ . It is known that, up to reversals, at most three different orders can be induced on a given  $F$  as  $l$  varies. It is also known that the families are of six different types, according to the number of orders and their interrelations. In this paper we study these types closely, focusing on their relations to the given set  $K$ , and on what happens as  $|F| \rightarrow \infty$ .

## 1 Introduction

Let  $F$  be a non-empty family of disjoint convex sets in the plane. Then a *transversal* of  $F$  is a line meeting all sets of  $F$ . If the line is oriented one gets a certain ordering of  $F$ , which is reversed if the orientation is changed. Such a pair of orderings is called a *geometric permutation* of  $F$  (abbreviated: a GP of  $F$ ). We describe a GP by listing the sets using one of the two orders. In 1957 Hadwiger [7] proved that if  $F$  is finite, then  $F$  admits a transversal if and only if  $F$  has an ordering,  $\prec$ , satisfying the following condition: whenever  $X \prec Y \prec Z$ ,  $\{X, Y, Z\}$  admits the GP  $(XYZ)$ . This result shows that once we know, for each triple of sets from  $F$ , which GPs it admits, we know whether  $F$  admits a transversal or not. No further knowledge of  $F$  is needed.

Using Hadwiger's theorem it is for instance easy to see that if each triple from  $F$  admits exactly one GP, while each quadruple of  $F$  has a transversal, then  $F$  has a transversal. One longstanding conjecture by Grünbaum [6] stated that if  $F$  consists of at least five pairwise disjoint translates of a given compact convex set  $K$ , and each quintuple from  $F$  has a transversal, then  $F$  has one. This conjecture was first proved in a weaker form (5 replaced by 128) by Katchalski [10] and then, as stated, by Tverberg [13]. In both proofs GPs played an important role.

Quite apart from their role as a tool, the GPs have turned out to be interesting objects of study in their own right. A natural question to ask is: Given  $|F|$ , how many different GPs can  $F$  have? Edelsbrunner and Sharir

[3] showed that the number of GPs is at most  $2|F| - 2$ , for  $|F| \geq 4$ , and Katchalski, Lewis, and Zaks [9] gave an example of a family admitting this number of GPs. In the present paper we study only what we shall call  $K$ -families. A  $K$ -family  $F$  consists of finitely many disjoint translates of a compact convex set  $K$ . We require  $K$  to be an *oval*, i.e. to have interior points, in order to avoid trivial exceptional cases in some of our statements.

In the case of a  $K$ -family  $F$ , the upper bound  $2|F| - 2$  can be replaced by 3, for  $|F| \geq 3$ . A refined version of this statement was proved by Asinowski, Holmsen, and Katchalski in [1]:

**Theorem 0.** *The set of GPs of a given  $K$ -family, if non-empty, must be of one of the following Types.*

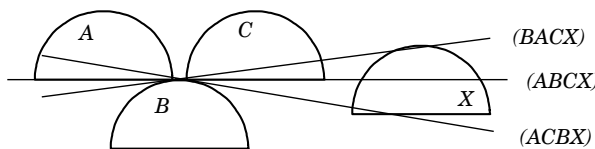
1.  $\{(W)\}$
2.  $\{(WABW'), (WBAW')\}$  (For this Type we assume that  $W$  and  $W'$  are not both empty)
3.  $\{(WABCW'), (WBCAW')\}$
4.  $\{(WABCXW'), (WBXACW')\}$
5.  $\{(WABCW'), (WACBW'), (WBACW')\}$
6.  $\{(WABXCW'), (WBAXCW'), (WACBXW')\}$

(To clarify the notation: If  $F$  is of Type 4, say, then  $F$  has two GPs, which can be written as  $(Y_1Y_2 \dots Y_mABCXY'_1 \dots Y'_n)$  and  $(Y_1Y_2 \dots Y_mBXACY'_1 \dots Y'_n)$ , where  $m \geq 0, n \geq 0$ .) Note that if  $F$  is of Type 3, with  $|F| = 3$ , then  $F$  can also be considered to be of Type 2. In all other cases,  $F$  has a unique Type. If  $F$  is of Type  $x$  and cardinality  $n$  we sometimes say that  $F$  is of Type  $x_n$ .

The main purpose of the present paper is to refine Theorem 0 in various ways. Firstly we want to see how it is affected if we specialize  $K$  and  $|F|$ . Secondly, we study what happens when  $|F| \rightarrow \infty$ . Apart from their intrinsic value, these results may also make the GPs a better tool for studies of transversal problems for  $K$ -families. For many of these problems deal with a special  $K$ , or a special class of  $K$ 's.

For a simple example, let  $K$  be a circular disc. Then the reader will easily construct a  $K$ -family of Type 5, with  $|F| = 3$ . A little experimentation will indicate that none exists for  $|F| = 4$ , and we shall in fact give a rigorous proof of that. If we instead let  $K$  be a semicircular disc, then figure 1 shows a  $K$ -family of Type 5 and size 4 and it will be clear that one can be found for any size.

The reader should be aware that this paper deals with a limited area within the larger field of transversals of, and in, all dimensions, as well as GPs and their higher-dimensional analogues. For more information consult i.e. [4], [5], [11].



**Fig. 1.** A  $K$ -family of Type 5 and size 4 (i.e. Type  $5_4$ ). The two non-horizontal lines touch two semidisks, the horizontal one touches three.

The material presented here originates in the Master’s theses of Asinowski [2] and Holmsen [8], who were supervised by, respectively, Katchalski and Tverberg. All the four authors participated in the revisions and expansions which resulted in the present paper.

## 2 Types 1 and 2

We start by introducing some basic definitions. Let  $K$  be a given oval, and  $x$  a given Type, and assume that there are arbitrarily large  $K$ -families of Type  $x$ . We then say that  $K$  is  $x$ -good. Let furthermore, for an  $x$ -good  $K$ ,  $F_1, F_2, \dots$  be  $K$ -families, where each family  $F_n$  is of Type  $x$  and has at least  $n$  members. We shall then say that the sequence  $\{F_n\}$  is  $x$ -good. (Note that, when  $m \neq n$ ,  $F_m$  bears no relation to  $F_n$ , except that both are  $K$ -families).

Let now  $\{F_n\}$  be an  $x$ -good  $K$ -sequence, and let  $\{t_n\}$  be a sequence of lines, where  $t_n$ , of slope  $s_n \in \mathbf{R} \cup +\infty$ , is a transversal for  $F_n$ . If  $\{s_n\}$  converges to a “number” in  $\mathbf{R} \cup \{+\infty, -\infty\}$ , we say that the corresponding direction is  $x$ -good for  $K$ . Note that if *one* sequence  $\{t_n\}$  shows  $x$ -goodness of  $K$ , then *any* sequence  $\{t'_n\}$ , where  $t'_n$  is a transversal for  $F_n$ , does the same, and gives the same  $x$ -good direction. For an easy argument shows that  $\lim_{n \rightarrow \infty} s'_n = \lim_{n \rightarrow \infty} s_n$ . (Since  $K$  has positive, finite area and finite diameter, there is a positive constant  $c_K$  such that some two sets of  $F_n$  are at least  $c_K \sqrt{n}$  apart, but then for some other positive constant  $c'_K$ , the angle between any lines meeting these two sets is at most  $c'_K / \sqrt{n}$ . This shows the equality of the two limits.)

It is clear that any oval  $K$  admits Type  $1_n$  for all  $n$ , and that every direction is 1-good for  $K$ . For Type 2 the situation is somewhat different.

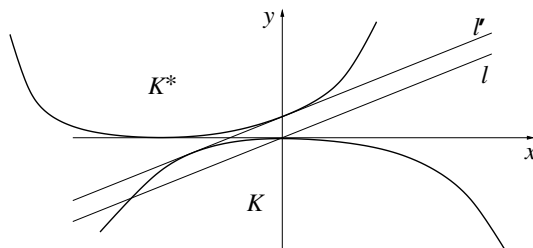
Here, and also later, the concept of an *extreme* tangent will be important, so we take a close look at it. For a given oval  $K$  most of its boundary points  $p$  will be smooth, i.e.  $K$  has exactly one tangent at  $p$  which is then declared extreme. The set of non-smooth boundary points is at most countable. If  $p$  is non-smooth, the tangents at  $p$  fill a double cone with apex at  $p$ . Then the boundary of that cone is the union of two tangents declared to be extreme at  $p$ . A little thought shows that we could have defined an extreme tangent at  $p$  as a tangent which is the limit of secants through  $p$ , to the curve  $\text{bdry}(K)$ . This will be useful later on. We now show the following theorem.

**Theorem 1.** *Let  $K$  be an oval.*

- a)  $K$  admits Type  $2_n$  for all  $n \geq 3$ .  
 b) A direction is 2-good for  $K$  if and only if  $K$  has an extreme tangent in that direction.

Note that a square has only two 2-good directions, because of b), while for a circular disc all directions are 2-good, also by b).

*Proof of Theorem 1.* For a) we choose a smooth boundary point  $p$  of  $K$  (we remind the reader that there are at most countably many non-smooth boundary points). Assume that  $p$  is not contained in a segment of the boundary of  $K$ . The case where  $p$  is contained in a boundary segment is treated similarly, and is left to the reader. It is no serious restriction to assume that  $K$  is in the halfplane  $y \leq 0$ , that  $p$  is the origin, and the  $x$ -axis the unique tangent. Consider a line  $l$  through  $p$ , with a small positive slope. Then  $l$  is not a tangent to  $K$ , but becomes one,  $l'$ , when lifted by the right amount. We now place a translate  $K^*$  of  $K$  in the unique position where it touches the negative  $x$ -axis and the part of  $l'$  in the halfplane  $y \geq 0$  (see figure 2). It is now clear that if the slope of  $l$  is small enough (dependent on  $K$  and  $n$ ), we can place  $n - 2$  further translates, touching the  $x$ -axis from above such that the  $x$ -axis induces  $(K^* K K_1 \dots K_{n-2})$  while  $l'$  induces  $(K K^* K_1 \dots K_{n-2})$  and no third GP exists.



**Fig. 2.** The  $x$ -axis and  $l'$  are common separating tangents for  $K$  and  $K'$

In order to prove the sufficiency part of b) we note that in the construction just given we did not use the smoothness at  $p$  to the full extent. If  $K$  has, say, a horizontal extreme tangent  $t$  at some point  $p$ , then if  $t$  is not a limit of ascending secant lines through  $p$ , it must be a limit of descending ones. In the latter case we may assume after reflection in the  $y$ -axis that  $K$  is located as in the proof of a), with the  $x$ -axis being the limit of ascending secant lines through  $p$ . Then the argument above is still valid.

For the proof of the necessity part of b) let, say, the horizontal direction be 2-good for  $K$  and for the 2-good  $K$ -sequence  $\{F_n\}$ . We have to prove that  $K$  has a horizontal extreme tangent, and so we assume that this is not the case.

In a segment in  $\text{bdry}(K)$  every point, except possibly the endpoints, is smooth. Our assumption thus implies that the two horizontal tangents to  $K$  meet  $K$  in single points  $p^+$  (at the top) and  $p^-$  (at the bottom). By non-extremeness  $K$  has two tangents, one of slope  $\epsilon > 0$  and one of slope  $-\epsilon$  at  $p^+$ , and also at  $p^-$ .

Consider now the  $K$ -family  $\{F_n\}$ , with GPs  $\pi_{1,n} = (W_n A_n B_n W'_n)$  and  $\pi_{2,n} = (W_n B_n A_n W'_n)$ . Let  $t_n$  be one of the two tangents to  $A_n$  and  $B_n$  which separate  $A_n$  from  $B_n$ . Then  $t_n$  is a transversal for  $F_n$ . For let  $C$  belong to  $F_n$  without meeting  $t_n$ . One of the closed halfplanes defined by  $t_n$ , say the one containing  $A_n$ , has  $C$  in its interior. Then  $\text{conv}(A_n \cup C)$  does not meet  $B_n$ , hence the GP  $(A_n B_n C)$  does not exist. But then one of  $\pi_{1,n}$  and  $\pi_{2,n}$  does not exist either.

With  $t_n$  being a transversal of  $F_n$ , we know from the second paragraph of this section that we can choose  $n$  such that the slope of  $t_n$  is less than  $\epsilon$  in absolute value. Let, say,  $A_n$  be in the lower one, and  $B_n$  in the upper one of the two closed halfplanes bounded by  $t_n$ , and let  $p_n^+$  be the top point of  $A_n$ . Then  $A_n$  is contained in the cone which has its apex in  $p_n^+$  and its left (right) boundary ray of slope  $\epsilon$  ( $-\epsilon$ ). Hence  $t_n$  can only meet  $A_n$ , which it does, in  $p_n^+$ . Similarly, for large  $n$ , the other separating tangent  $t'_n$  also contains  $p_n^+$ , and so  $t_n \cap t'_n \in A_n$ . By symmetry,  $t_n \cap t'_n \in B_n$ , which leads to the final contradiction  $A_n \cap B_n \neq \emptyset$ .  $\square$

### 3 On parallelograms

Parallelograms behave in a nice and simple way, as expressed by the following.

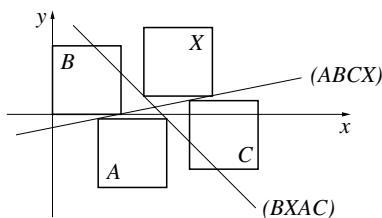
**Theorem 2.** *Let  $P$  be a parallelogram.*

- a)  *$P$  admits Type  $x_n$  for every  $n \geq x$ ,  $x = 1, 2, 3, 4$ .*
- b)  *$P$  does not admit Type 5 or 6.*
- c) *If an oval  $K$  does not admit Type  $5_3$  then  $K$  is a parallelogram.*
- d) *For  $2 \leq x \leq 4$   $P$  has just two  $x$ -good directions.*
- e) *If an oval  $K$  has just two 2-good directions, then  $K$  is a parallelogram.*

*Proof of Theorem 2.* After an affine transformation, which does not affect transversal properties and GPs, we may assume  $P$  to be a square. We first prove a), b), and d).

For b), note that any two parallel rectangles can be strictly separated by a horizontal or vertical line (or both). Therefore all ascending transversals define the same GP, and similarly for all descending transversals, so a family of disjoint parallel rectangles can have at most 2 GPs, thus Types 5 and 6 are impossible.

Here a) is already known for the cases  $x = 1$  and  $x = 2$ . The remaining two cases are dealt with by the following construction.



**Fig. 3.** The two induced GPs

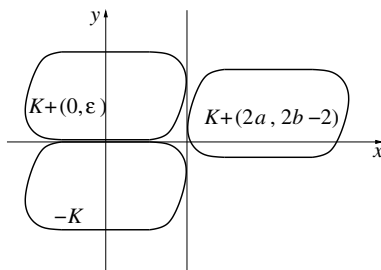
Let  $\epsilon$  be a number in  $(0, 1/3)$  and let  $A$ ,  $B$ ,  $C$ , and  $X$  be squares, with axis-parallel sides of length 3 and SW (= South West = lower left) corners in  $(2, -3 - \epsilon)$ ,  $(0, 0)$ ,  $(6, -3 + 3\epsilon)$ , and  $(4, 4\epsilon)$ , respectively. Then the upper non-separating common tangent to  $A$  and  $C$  has slope  $\epsilon$  and induces  $(ABCX)$ , while the separating common tangent to  $X$  and  $A$  has slope  $-5\epsilon$  and induces  $(BXAC)$ . By *b*), there cannot be a third GP. See figure 3 for an illustration.

One can now place, for any  $n \geq 4$ ,  $n - 4$  further squares  $S_1, S_2, \dots, S_{n-4}$  with the SW corner of  $S_i$  in  $(4i+10, -1)$ . Then the family  $\{A, B, C, X, S_1, \dots, S_{n-4}\}$  is clearly of Type  $4_n$  provided  $\epsilon$  has been chosen small enough. And the family  $\{A, B, X, S_1, \dots, S_{n-4}\}$  is of Type  $3_{(n-1)}$ , as the GP  $(BAX)$  does not exist.

Clearly *d*) follows from Theorem 1.

Here we prove *c*) only partly, for centrally symmetric  $K$ . The general case will be dealt with in section 5. Let  $K$  be centrally symmetric, but not a parallelogram. We start with the case when  $K$  has a segment in its boundary. We can assume, after an affine transformation, that  $K$  has its center in  $(0, 1)$ , while the  $x$ -axis intersects  $K$  in the segment  $[(-1, 0), (1, 0)]$ . This segment is also the top of  $-K$  (the reflection of  $K$  in the origin). Because of the central symmetry,  $-K$  is also a translate of  $K$ , in fact  $-K = K + (0, -2)$ . The two vertical tangents to  $K$ , given by  $x = \pm a$ ,  $a > 0$ , are also tangents to  $-K$ . Here  $a > 1$ , as  $K$  is not a square. If the right tangent intersects  $K$  and  $-K$  in single points  $(a, b)$  and  $(a, b - 2)$  we get, for a sufficiently small  $\epsilon > 0$ , the three desired GPs for  $K + (0, \epsilon)$ ,  $-K$  and  $K + (2a, 2b - 2)$ . See figure 4, below, for an illustration. If, however, the intersection mentioned is a segment, we choose an almost vertical tangent  $t$  to  $K$ , with a negative slope, which meets  $K$  in a single point  $p$  close to, or on the segment. Now push  $-K$  rightwards until it touches  $t$  and place a third translate so that it is separated by  $t$  from  $K$  and (the displaced)  $-K$ , while it touches  $t$  at the intersection of  $t$  and the  $x$ -axis. Then again the three desired GPs exist, after  $K$  is lifted a little.

If  $K$  is strictly convex (i.e the boundary of  $K$  contains no proper segment), then  $K$  and  $-K$  can be assumed to meet in the smooth point  $(0, 0)$  with the  $x$ -axis as the common tangent, and  $(0, 1)$  as the center of  $K$ . Let  $p$  and  $q = (q_1, q_2)$  be the leftmost and rightmost points of  $K$ , and translate  $K$  so that  $p$  moves to  $(q_1, 0)$ . Then  $K + (0, \epsilon)$ ,  $-K$ , and the translated  $K$  will have the desired GPs for all small  $\epsilon > 0$ .



**Fig. 4.**  $K$  is not a parallelogram  $\Rightarrow K$  admits Type  $5_3$ .

We finally prove *e*). We have to find three 2-good directions, for any given oval  $K$ , not a parallelogram. Now  $\text{bdry}(K)$  is either a polygon, or infinitely many non parallel segments, or it contains an arc with no segments. Each of these three cases are easily dealt with using Theorem 1 *b*).  $\square$

#### 4 On ellipses

By an *ellipse* we mean an oval with boundary an ellipse in the usual sense. Ellipses, like parallelograms, have an easily describable behavior with regard to GPs, as seen from the following.

**Theorem 3.** *Let  $E$  be an ellipse.*

- a)  $E$  is 1-good and 2-good for all directions.*
- b)  $E$  is not  $x$ -good for  $x > 2$ .*
- c)  $E$  does not admit the Types 4 and 6.*
- d)  $E$  admits Type  $3_n$  or  $5_n$  only for  $n = 3$ .*

**Remark.** Statement *b*) was also proved by Smorodinsky, Mitchell, and Sharir in [12].

*Proof of Theorem 3.* Clearly *a*) holds (and is a consequence of Theorem 1). Obviously *b*) will follow from *c*) and *d*), while corollary 1 in section 7 will prove *c*). As to *d*) we first observe that if  $F_n$  is a  $K$ - $n$ -family of Type  $3_n$ , with  $n \geq 4$ ,  $W_n$  or  $W'_n$  must be non-empty. If  $W_n$  is non-empty then  $F_n$  has a subfamily  $\{A_n, B_n, C_n, Y_n\}$  which allows (at least) the GPs  $(Y_n A_n B_n C_n)$  and  $(Y_n B_n C_n A_n)$ . If  $W'_n$  is non-empty, there will be GPs  $(A_n B_n C_n Y'_n)$  and  $(B_n C_n A_n Y'_n)$  or, differently expressed,  $(Y'_n A_n C_n B_n)$  and  $(Y'_n C_n B_n A_n)$ . If  $F_n$  is of Type  $5_n$ , we can ignore the first GP  $(W_n A_n B_n C_n W'_n)$  and apply the same reasoning. Thus *d*) will follow from the lemma below.  $\square$

**Lemma 1.** *When  $K$  is a circular disc, a  $K$ -family can not have both of the GPs  $(ABCD)$  and  $(ACDB)$ .*

*Proof of Lemma 1.* We prove the lemma by contradiction, so assume we have  $(ABCD)$  and  $(ACDB)$ . Now we may assume that  $(ABCD)$  is a *critical* GP, i.e. there is a unique line inducing  $(ABCD)$ . This assumption is justified by a shrinking argument often used in the study of transversal problems. To shrink an oval  $A$  we first chose as shrinking center a point  $a_0$  in  $A$ . The shrinking process will then consist in reducing  $A$  continuously, by letting a parameter  $\lambda$  vary from 1 downwards and letting for every  $\lambda$   $A_\lambda$  be the set  $\{a_0 + \lambda(a - a_0), a \in A\} = \lambda A + (1 - \lambda)a_0$ . If  $F$  is a  $K$ -family  $\{K + v_i, i \in I\}$  and the sets  $K + v_i$  are shrunk simultaneously with centers  $k_i + v_i$ , where  $k_i \in K$  we get for every  $\lambda$  a  $\lambda K$ -family  $F_\lambda = \{\lambda(K + v_i) + (1 - \lambda)(k_i + v_i), i \in I\} = \{\lambda k_i + v_i + (1 - \lambda)k_i, i \in I\}$ .

In the present case we first shrink the discs around their centers until one of the two GPs of  $F_\lambda$  becomes critical. Thus after renaming the discs we may assume one of the GPs of  $\{A, B, C, D\}$  to be critical. If the critical GP is  $(ACDB)$  we shrink further, but now with centers chosen on the transversal which induces  $(ACDB)$ . At some stage also the other GP becomes critical, and so our assumption is justified.

Let  $l$  be the unique line that determines  $(ABCD)$ . We may assume that  $l$  is horizontal, and that we meet the sets in the order  $ABCD$  when we traverse  $l$  from left to right.

Now the reader will easily verify that a GP of a family is critical exactly when it induces a critical GP for some subfamily of size three. From now on we can clearly assume that two sets of our family lie below  $l$ . In our case  $(BCD)$  cannot be a critical GP. For then there will exist a vertical line that separates  $D$  strictly from  $A$  and  $B$ , so that we cannot have  $(ADB)$ , and hence not  $(ACDB)$ .

We now make an observation, more general than needed here, but we will also use it later. Let a  $K$ -family  $F$ , where  $K$  need not be a disc, be given. Then any directed transversal  $t$  meets  $F$  first in a translate, say  $A$ , which is not in the convex hull of the others (which is the same as saying that if one picks a point of  $A$ , that point is not in the convex hull of the corresponding points of the other translates). For assume the falsity of the statement on  $t$ . By Carathéodory's theorem this means that, say,  $A \subset \text{conv}(B \cup C)$  or  $A \subset \text{conv}(B \cup C \cup D)$  where  $B, C, D$  are other members of  $F$ . In the first case  $t$  must induce  $BAC$  so it cannot meet  $A$  first. If  $A \subset \text{conv}(B \cup C \cup D)$ , then  $t$  must enter, say,  $\text{conv}(B \cup C)$  before it can get to  $A$ , and then leave  $A$  before it has left  $\text{conv}(B \cup C)$ . But clearly the part of  $t$  before  $A$ , together with  $A$ , separates  $B$  from  $C$  in  $\text{conv}(B \cup C)$ , so that  $t$  cannot meet both  $B$  and  $C$  after  $A$ .

By the observation just made it follows that if  $(ACD)$  is critical then the centers of the discs must form a convex quadrilateral with cyclic order  $ABCD$ , but this violates the fact that we have  $(ACDB)$ , as shown in section 5 of [13].

Now consider the case where  $(ABD)$  is the critical GP. The convex hull of the centers must be a triangle, or else we end up with a violation of the

cyclic order, as before. Let  $m$  be the vertical line which is tangent to the right side of  $B$ . Clearly  $D$  cannot lie to the left of  $m$ , and if  $D$  lies to the right of  $m$ ,  $D$  would not meet  $\text{conv}(A \cup B)$ . Thus  $m$  must meet  $D$ . We obtain a contradiction as follows. Move  $D$  away from  $A$  in the horizontal direction, until it becomes tangent to  $m$ . Note that while doing this, the disjointness of the discs will be preserved. Now  $B$  and  $D$  lie in opposite quadrants defined by  $l$  and  $m$ , and  $C \cap l$  lies between  $B \cap l$  and  $D \cap l$ . It is easy to see that this contradicts the disjointness of the discs.

Finally, consider the case where  $(ABC)$  is the critical GP. The convex hull of the centers must then be a quadrilateral. We also see that the cyclic order of the centers must be  $ABDC$ . We now move  $D$  away from  $A$ ,  $B$ , and  $C$ , in the direction of  $l$ . This will cause  $(ADB)$  (and thus  $(ACDB)$ ) to become a critical GP. But the situation we have described cannot occur. To see this, assume without loss of generality that the discs are of diameter 1 and let  $a, b, c, d$  denote the centers of  $A, B, C, D$ , respectively. Further let  $(x_z, y_z)$  denote the coordinates of the center of the disc  $Z$ . Assume that  $a$  and  $c$  lie on the  $x$ -axis, with  $c$  at the origin and  $x_a < -1$ . Since  $(ABC)$  is a critical GP it follows that  $b$  lies on the line  $y = 1$ . Thus,  $d$  must lie in the first quadrant, below the line  $y = 1$ , and by the disjointness of the discs  $d$  must lie outside the closed unit circle centered at the origin. Let  $m$  be the line through  $a$  and  $b$ . Since  $(ADB)$  is a critical GP, the distance from  $d$  to  $m$  must equal 1, and the distance from  $c$  to  $m$  must be less than 1. We therefore have  $x_a < -1 < x_b < 0$ . Let  $m'$  be the line parallel to  $m$  which goes through the point  $d$ . Let  $b'$  be the orthogonal projection of  $b$  on the line  $m'$ . The orthogonal projections of the centers on the line  $m'$  must have the order  $acdb$ . This implies that  $d$  must lie below  $b'$  on the line  $m'$ . The situation is illustrated in figure 5.

Figure 5 indicates that the point  $b'$  has distance less than 1 from  $c$ . In order to see this, we find that

$$b' = \left( x_b + \frac{1}{\sqrt{1 + (x_a - x_b)^2}}, 1 + \frac{x_a - x_b}{\sqrt{1 + (x_a - x_b)^2}} \right)$$

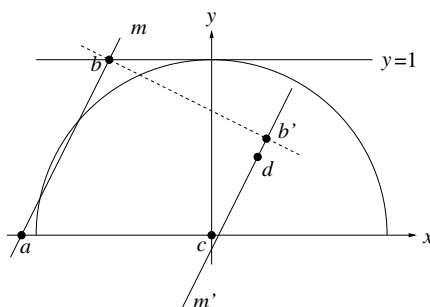


Fig. 5. The centers of the discs.

and thus

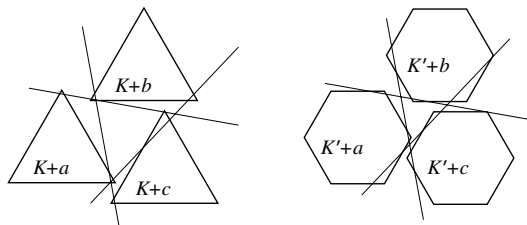
$$|cb'|^2 = x_b^2 + 2 + \frac{2x_a}{\sqrt{x_a^2 + x_b^2 - 2x_ax_b + 1}}$$

For a fixed slope of  $m$ ,  $|cb'|$  increases when  $m$  is moved to the right. Thus we only have to consider the limit cases when  $b = (0, 1)$  or  $a = (-1, 0)$ . Easy calculations show that the distance 1 is assumed only for  $a = (-1, 0)$ ,  $b = (-1, 1)$ . Thus  $b'$  belongs to the open unit disc centered at the origin, and so does  $d$ . But this contradicts the disjointness of the discs  $C$  and  $D$ .  $\square$

## 5 On Minkowski symmetrization

This classical symmetrization consists in associating with a convex body  $K$  another, centrally symmetric one  $K'$ , defined by  $K' = \frac{1}{2}K - \frac{1}{2}K$ . It has turned out to be useful also in transversal problems for translates (cf. [13], for instance). Let  $F = \{K + v_i, i \in I\}$  be a family of the type studied in this paper. Then  $F' = \{K' + v_i, i \in I\}$  is easily seen to be a family of the same type. Assume now that  $F$  has, say, a horizontal transversal  $t$ , and that  $K$  has height 1, i.e. the minimal horizontal strip containing  $K$  has width 1. Then some horizontal strip of width 1 contains all the  $v_i$ . Conversely, if a horizontal strip of width 1 covers all the  $v_i$ ,  $F$  has a horizontal transversal. Now, as is easily seen,  $F'$  also has height 1. This means in view of the preceding arguments, that  $F'$  has a horizontal transversal if and only if  $F$  has one. The same applies to every direction, of course.

What is important to note here, as we are concerned with GPs, is that when a GP is induced by, say, a horizontal transversal of  $F$ , then any horizontal transversal of  $F'$  induces the corresponding GP of  $F'$  (recall here that all parallel transversals induce the same GP on a given  $F$ ). For  $F$  can be transformed continuously into  $F'$  via the families  $F_s = \{((1-s)K + sK') + v_i, i \in I\}$ ,  $s \in [0, 1]$ . Straightforward calculations show that  $F_s$  is a family of the type under study and that  $(1-s)K + sK'$  has the same height as  $K$ . By continuity it follows that  $F_0 (= F)$  has the same ‘‘horizontal’’ GPs as  $F_1 (= F')$ . The same holds, of course, for any direction. An example of this is illustrated in figure 6.



**Fig. 6.**  $\{K + v, v \in \{a, b, c\}\}$  and  $\{K' + v, v \in \{a, b, c\}\}$  have the same GPs in the same directions.

We can now complete the proof of Theorem 2*c*). We proved that if the oval  $K$  is centrally symmetric and does not admit Type 5<sub>3</sub>, then  $K$  is a parallelogram. If  $K$  is not centrally symmetric and does not admit Type 5<sub>3</sub>, we know from the considerations above that the symmetrization  $K'$  does not admit Type 5<sub>3</sub>, and thus is a parallelogram. Hence  $K'$  and, accordingly,  $K$  have only two 2-good directions. But then, by Theorem 2*e*),  $K$  is a parallelogram. We shall make more use of symmetry later.

### 6 Type 3

Every oval admits Type 2<sub>3</sub> and therefore Type 3<sub>3</sub>. But by Theorem 3 an ellipse does not admit Type 3 <sub>$n$</sub>  for any  $n \geq 4$ . Now, in view of section 5, we can say even more; If  $K$  is affinely equivalent to a set of constant width (i.e. all its orthogonal projections have the same length),  $K$  does not admit Type 3 <sub>$n$</sub>  for  $n \geq 4$ . For if  $K$  is a set of constant width then its symmetrization  $K'$  is known to be a disc, which does not admit Type 3 <sub>$n$</sub> .

We shall now deduce a necessary condition for a centrally symmetric  $K$  to be 3-good for, say, the horizontal direction, and then see how it can be sharpened to become both necessary and sufficient.

Let the 3-good  $K$ -sequence  $\{F_n\}$  be given, so that we have, for  $n = 1, 2, \dots$ ,  $(W_n A_n B_n C_n W'_n)$  and  $(W_n B_n C_n A_n W'_n)$ . Let the good direction for  $\{F_n\}$  be horizontal. Consider an  $F_n$ . If we translate each member of it by the same vector  $v_n$ , the new family  $F'_n$  is still a  $K$ -family. If we do this for every  $n$  ( $v_n$  may depend on  $n$ ) the sequence  $\{F'_n\}$  is a 3-good sequence, with the horizontal direction being good. This means that if we choose some fixed translate  $A$  of  $K$ , then we may assume that  $A$  belongs to every  $F_n$  and that the two GPs of  $F_n$  are  $(W_n A B_n C_n W'_n)$  and  $(W_n B_n C_n A W'_n)$ .

We now ignore the  $C_n$ 's. From Theorem 1 and parts of its proof we find that the common separating tangents  $t_n$  and  $t'_n$  to  $A$  and  $B_n$  both converge to, say, the upper horizontal tangent to  $A$ . (Note: the assumed falsity of Theorem 1 does not enter into those parts we use here). We assume this tangent to be the  $x$ -axis.

Let  $a_n$  ( $a'_n$ ) be the point on  $t_n \cap A$  ( $t'_n \cap A$ ) nearest to  $t_n \cap t'_n$ . Then the distance from  $t_n \cap t'_n$  to  $\text{conv}(\{a_n, a'_n\}) \subset A$  converges to 0, because  $t_n$  and  $t'_n$  converge to the  $x$ -axis. Therefore  $\{t_n \cap t'_n\}$  has an accumulation point in  $A$ , so that after passing to a subsequence of  $\{F_n\}$ , we may assume  $\{t_n \cap t'_n\}$  to converge to a point  $p$  in  $A$ . Considering the similarly defined points  $b_n$  and  $b'_n$ , we find that  $p$  is an accumulation point for a sequence  $\{b''_n\}$ , with  $b''_n \in B_n$ . Thus, after passing once more to a subsequence, we can assume that  $\{B_n\}$  converges to a set  $B$ .  $B$  is of course a translate of  $K$  and  $A \cap B$  is either the point  $p$  or a segment containing  $p$ . We can assume  $p$  to be the origin.

If we now treat the "new"  $F_n$  in the same way as the original one, but ignore  $B_n$  this time, we can add the further assumption that  $\{C_n\}$  converges to  $C$ , a translate of  $K$ , and that  $A \cap C$  is a point or a segment in a horizontal

tangent to  $A$ . That tangent can not be the lower one, as for large  $n$  the almost horizontal transversals to  $F_n$  would then be impossible. Thus the tangent is the  $x$ -axis, and so  $C$  is a horizontal translate of  $B$ .

We now draw an important conclusion: The top (and hence the bottom, by central symmetry) of  $K$  must be flat. For if it is a single point, the bottom point of  $B$  must coincide with the top point of  $A$ , since  $B$  meets  $A$ . But the same holds for  $C$  vs.  $A$ , and thus  $B$  and  $C$  must be equal, which contradicts, for large  $n$ ,  $B_n \cap C_n = \emptyset$ .

Let the length of the top segment of  $K$  be 1, say. Assume that  $B$  is to the left of  $C$ . The intersection of  $B$  and  $C$  need not be empty, but it is clear that it can be at most 1-dimensional. This again implies that a horizontal segment in  $B$  ( $C$ ) can meet  $C$  ( $B$ ) in at most one point. Consider now  $B_1$  and  $C_1$ , obtained by translating  $B$  to the left and  $C$  to the right until they both meet  $A$  in just one point. Then  $C_1 = B_1 + (2, 0)$ . Since the longest horizontal chord of  $B_1$  meets  $C_1$  in at most one point, it follows that its length is at most 2. Here equality can only happen if  $B = B_1$  and  $C = C_1$ . It is easy to see which are the longest horizontal chords of  $K$ : By convexity and central symmetry they form a parallelogram (possibly degenerate), centrally symmetric about the symmetry center of  $K$ . In particular the horizontal central chord is a maximal one, and so we have: If  $K$  is 3-good for a given direction, then its two boundary segments in that direction must be at least half as long as the corresponding central chord.

The necessary condition just found, is almost sufficient, in the following sense: If a centrally symmetric  $K$  has, say, a horizontal segment in its boundary, with the horizontal central chord of length strictly less than twice the length of the segment, then  $K$  is 3-good. For then we can place two translates  $B$  and  $C$  on top of a third,  $A$ , so that  $B$  and  $C$  are disjoint, but both have 1-dimensional intersection with  $A$ . With  $B_n = B + (0, n^{-1})$ ,  $C_n = C + (0, n^{-1})$  we have for all large  $n$  the almost horizontal transversals inducing  $AB_nC_n$  and  $B_nC_nA$  which demonstrate 3-goodness of  $K$  (cf. the sufficiency part of Theorem 1).

If however the length of the parallel central chord is exactly twice that of the segment, an additional condition enters: At least one of the endpoints of the segment must be smooth.

In order to see the sufficiency of this condition we proceed as in the preceding paragraph, but now  $B$  and  $C$  touch and they both meet  $A$  in just one point. If, say, the left endpoint of the top segment of  $K$  is smooth, we saw in the proof of Theorem 1 how to get a  $B^*$  slightly further to the left than  $B$ , such that the  $x$ -axis induces  $B^*A$ , while a small-sloped tangent to  $A$  and  $B^*$  induces  $(AB^*)$ . As  $B^*$  is further to the left than  $B$ , we can move  $C$  a little to the left to get  $C^*$ , disjoint from  $B^*$ , but having 1-dimensional intersection with  $A$ . Lifting  $C^*$  a little we get  $C^{**}$ , disjoint also from  $A$ , and now it is clear that the GPs  $(AB^*C^{**})$  and  $(B^*C^{**}A)$  exist and are induced by almost horizontal transversals. Note that  $A$  is disjoint from  $\text{conv}(B^* \cup C^{**})$  so the GP  $B^*AC^{**}$  does not exist. Thus the construction just shows 3-goodness of

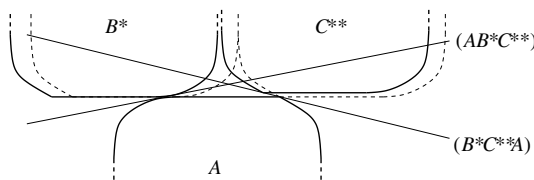


Fig. 7. The lines induce the two GPs of Type 3.

$K$ . See figure 7 for an illustration. We note, for later use, that the same  $K$  is also 5-good. In fact, we could have created a  $B^{**}$  by translating  $B^*$  very little downwards along the small-sloped tangent mentioned above before we lifted  $C^*$ , and thus obtained also  $(B^{**}AC^{**})$ .

It remains to check that if the base and top of  $K$  have both endpoints non-smooth, and has length half that of the horizontal central chord, then the horizontal direction is not 3-good for  $K$ .

In this situation  $B$  and  $C$  intersect, either in just the common end,  $e$ , of their horizontal central chords, or in a non-horizontal segment having  $e$  as its midpoint. Now, for any 3-good  $\{F_n\}$ , with  $A_n$  constant, ( $= A$ ), and  $B_n$  converging to  $B$ ,  $C_n$  to  $C$ ,  $B_n$  has to contain  $e$  for large  $n$ . For consider the non-horizontal extreme tangent to  $A$  at the origin, which we after an affine transformation may assume to be the  $y$ -axis. Thus  $A$  lies in the fourth quadrant with the axes as extreme tangents at the origin. Then  $B_n$  cannot lie in the closed left half-plane for large  $n$ , which forces it to meet the open right half-plane, and thus to meet the open first quadrant. Thus  $B_n$  is obtained by translating  $B$  a distance  $o(1)$  upwards and rightwards.

Now, because of central symmetry,  $B = -A$  and it is then clear that the point  $e$  is in the second open quadrant, or on the positive  $y$ -axis, and in the latter case the segment from  $e$  to the origin is of course contained in  $B$ . In the former case all the tangents to  $B$  at  $e$  has negative slope. In both cases  $B$  contains a rectangle with its NE corner at  $e$ , which means that for large  $n$ ,  $B_n$  will contain  $e$ .

By symmetry (not central!) we get a contradiction: Also  $C_n$  contains  $e$  for large  $n$ .

Some definitions are needed to formulate Theorem 4. Let a *side* of an oval  $K$  be the intersection of  $K$  and a tangent to  $K$ , of positive length. If  $K$  is centrally symmetric and the length of a side is at least half that of the parallel central chord, the side is said to be *long*. If the length exceeds half the chord length, then the side is *very long*. Then we have

**Theorem 4.** *Let  $K$  be a centrally symmetric oval.*

- a)  $K$  admits Type 3.
- b)  $K$  is 3-good for a direction if and only if the two corresponding sides are either very long, or long (but not very long) with at least one smooth endpoint.

*c)  $K$  has at most two 3-good directions.*

Here *a)* and *b)* have just been proved, while *c)* will follow from the following technical lemma:

**Lemma 2.** *A centrally symmetric  $K$  has at most three pairs of long sides. If it has three pairs, it is an affine image of a hexagon  $H_a$  defined by  $0 \leq x \leq a$ ,  $0 \leq y \leq a$ ,  $|y - x| \leq 1$ , with  $a \geq 2$ .  $H_a$  has at most one pair of very long sides.*

Lemma 2 will imply *c)*, in view of *b)*.

*Proof of Lemma 2.* Consider first the case when  $K$  is a hexagon, with all sides long. After an affine transformation we can assume its vertices to be  $(0, 1)$ ,  $(0, 0)$ ,  $(1, 0)$ ,  $(a, b)$ ,  $(a, b + 1)$ , and  $(a - 1, b + 1)$ , with  $0 < b \leq a - 1 > 0$ . If  $K$  has only two very long sides we choose the transformation such that  $(0, 0)$  is not on any of them. The line through  $(0, 0)$ , parallel to the side  $[(1, 0), (a, b)]$ , meets the side  $[(a, b), (a, b + 1)]$  in the point  $(a, ab/(a - 1))$ . Now the chord  $[(0, 0), (a, ab/(a - 1))]$  is not longer than the parallel central chord, which is at most twice as long as the parallel side  $[(1, 0), (a, b)]$ . Thus  $a - 0 \leq 2(a - 1)$ , i.e.  $a \geq 2$ .

Consider now the horizontal chord  $[(0, b/(a - 1)), (2, b/(a - 1))]$ . One of its endpoints must be a vertex, for if not, horizontal chords slightly above it will have lengths  $> 2$ , which is a contradiction. If  $(0, b/(a - 1))$  is a vertex,  $b = a - 1$  and the lemma holds, as  $K = H_a$  and the horizontal and vertical sides are not very long. If  $(2, b/(a - 1))$  is a vertex, then  $a = 2$ . If now  $b = 1$ , then  $K = H_2$ , which is an affine image of the regular hexagon, for which the lemma clearly holds. If  $a = 2$ , but  $b < 1$ , all sides are long, only the vertical ones are very long, and so we have not chosen the right affine transformation.

In the case when  $K$  is not a hexagon we assume the center of symmetry to be in the origin. If the lemma does not hold choose six long sides  $\rho, \sigma, \tau, -\rho, -\sigma, -\tau$ , with the stated order corresponding to the one along  $\text{bdry}(K)$ . Let  $K_+ = \text{conv}(\rho \cup \sigma \cup \tau)$ ,  $K_- = -K_+$ . As  $K$  is not a hexagon we may assume that  $\rho \cap (-\tau) = (-\rho) \cap \tau = \emptyset$ .  $K_+$  is a four-, five-, or six-sided polygon, with only one side  $\alpha$  that meets  $\rho$  and  $\tau$  but not  $\sigma$ . We translate  $K_+$  so that the midpoint of that side goes to  $(0, 0)$ . After a similar treatment of  $K_-$ , the union of the two translates will be a centrally symmetric oval  $K_1$ . The sides  $\pm\rho_1, \pm\sigma_1$ , and  $\pm\tau_1$  (the translates of  $\pm\rho, \pm\sigma, \pm\tau$ ) are long sides of  $K_1$ .  $\pm\rho_1$  and  $\pm\tau_1$  are in fact very long, while  $\pm\sigma_1$  are very long unless  $\pm\sigma_1$  are parallel to  $\alpha$ .  $K_1$  is a  $2p$ -gon with at least four very long sides and  $p \leq 5$ . After  $p - 3$  further applications of the procedure just described, we get a hexagon with at least four very long sides. But that has been proved impossible.  $\square$

## 7 Type 4

Any oval  $K$  admits Types 1, 2, and 3, but it may not admit Type 4. One property, useful for the discussion of which sets admit Type 4 is given by

**Lemma 3.** *If an oval  $K$  admits Type 4, then there are four translates of  $K$ , and two transversals, demonstrating this, which are tangents to all four translates.*

*Proof of Lemma 3.* We first note a consequence of the general observation made during the proof of Lemma 1: If  $K$  admits Type 4, so that we have translates of  $K$  admitting  $(ABCX)$  and  $(BXAC)$ , then the translates are convexly independent, as each translate is at an end of one GP. Thus we can speak of their cyclic order.

We now shrink  $A, B, C$ , and  $X$  towards their circumcenters until one of the GPs becomes critical (cf. the proof of Lemma 1). Let  $a, b, c$ , and  $x$  be points on the critical transversal, in  $A, B, C$ , and  $X$ , respectively, and then shrink further with  $a, b, c$ , and  $x$  as centers of contraction, until the other GP also becomes critical. Now observe that the set of GPs allow the name changes  $A \rightarrow B \rightarrow X \rightarrow C \rightarrow A$ ;  $A \leftrightarrow X, B \leftrightarrow C$ ;  $A \rightarrow C \rightarrow X \rightarrow B \rightarrow A$ . This fact allows the assumption that  $(ABCX)$  is induced by a transversal which touches, and separates,  $X$  and  $C$ , while it either touches  $A$  separating it from  $C$ , or touches  $B$ , separating it from  $C$ . Here only the second case can occur, for in the first case it is clear that the cyclic order of the translates is  $ABCX$ . This is however incompatible with  $(BXAC)$ , as shown in [13], section 5.

The argument just given shows, by symmetry, that for  $(BXAC)$ , either  $(XAC)$  or  $(BXA)$  must be the critical induced GP. In the second case the relabeling  $A \rightarrow B \rightarrow X \rightarrow C \rightarrow A$  shows that we can assume that we are in the first case. Thus the full GPs are  $(ABCX)$  and  $(BXAC)$ , while the critical shorter ones are  $(BCX)$  and  $(XAC)$ .

We can now assume that  $(ABCX)$  is induced by the  $x$ -axis and  $(BXAC)$  by the  $y$ -axis, with  $C$  being in the first quadrant. Then  $X$  must be in the fourth quadrant, while  $A$  touches the positive  $y$ -axis from the left and  $B$  touches the negative  $x$ -axis from below. Let  $A^*$  be the upwards translate of  $A$ , touching the  $x$ -axis, and  $B^*$  the leftwards translate of  $B$ , touching the  $y$ -axis. Clearly  $A^* \cap X = B^* \cap X = B^* \cap C = \emptyset$ . The couple  $(A^*, B^*)$  is a translate of  $(C, X)$  and so  $A^* \cap B^* = \emptyset$ .  $(A^*, C)$  is a translate of  $(B^*, X)$  and so  $A^* \cap C = \emptyset$ . The  $x$ -axis induces  $A^*B^*CX$ , the  $y$ -axis  $B^*XA^*C$ , and so Lemma 3 is proved.  $\square$

**Corollary 1.** *An ellipse does not admit Types 4 and 6.*

*Proof of Corollary 1.* It suffices to prove that a circle does not admit Type 4. This follows from Lemma 3, for when two lines touch the circles  $A, B, C$ , and  $X$ , and one of them induces  $ABCX$ , the other will induce  $ACBX$ .  $\square$

It seems natural to ask the following question: Which  $K$ s do not admit Type 4? We know, using Corollary 1, that a  $K$  affinely equivalent to a set of constant width does not admit Type 4, as the Minkowski symmetrization of the latter is a disc. But there are other examples, too, for instance all regular  $n$ -gons with  $n \not\equiv 0 \pmod{4}$ .

As to 4-goodness, we have the following theorem.

**Theorem 5.** *A given centrally symmetric oval  $K$  is 4-good if and only if it is 3-good.*

*Proof of Theorem 5.* We start with the “only if” part. Let  $K$  be 4-good, and let  $\{F_n\}$  be a 4-good sequence for  $K$ . Thus we have GPs  $(W_n A_n B_n C_n X_n W'_n)$  and  $(W_n B_n X_n A_n C_n W'_n)$ ,  $n = 1, 2, \dots$ . If we ignore the sets  $B_n$  we get a sequence  $\{F_n^*\}$  where, for every  $n$ ,  $F_n^*$  is either of Type 3 or admits a further GP. But when we deduced the necessary condition for  $K$  being of Type 3, we did not use fully the fact that the  $n$ 'th  $K$ -family was of Type 3.: We just used the fact that there was for every  $n$ , a  $K$ -family having two GPs of the Type used to define Type 2, and did not care about a possible third one. Thus our  $\{F_n\}$  shows that  $K$  satisfies the necessary condition for  $K$  being of Type 3. But that condition was also sufficient, and thus  $K$  is 3-good if it is 4-good.

Let now  $K$  be 3-good. By Theorem 4,  $K$  has one or two pairs of long sides corresponding to good directions. Consider such a pair, assumed to be horizontal. Let  $K^+$ ,  $K^-$  denote, respectively, the top and bottom segment of  $K$ , and use a similar notation for translates of  $K$ . We place four translates of  $K$ :  $A$ ,  $B$ ,  $C$ , and  $X$ , such that  $B^-$ ,  $A^+$ ,  $X^-$ , and  $C^+$  are, say, on the  $x$ -axis, with say,  $B^- \cap A^+ = (-1, 0)$ ,  $A^+ \cap X^- = (0, 0)$ ,  $X^- \cap C^+ = (1, 0)$ . We now have to move the translates a little, in order to make them disjoint, and having the desired GPs induced by almost horizontal transversals.

The easier case is when  $K^+$  is very long. After small displacements of  $B$ ,  $X$ , and  $C$ , each of  $B \cap A$ ,  $A \cap X$ , and  $X \cap C$  is a small horizontal segment. We then lift  $X$  and  $C$  by the same small amount. Now the line through the midpoints of  $A \cap B$  and  $X \cap C$  induces  $(\text{int}(A) \text{ int}(B) \text{ int}(C) \text{ int}(X))$ , while the falling, separating, common tangent for  $X$  and  $A$  induces  $(\text{int}(B) X A \text{ int}(C))$ . (Note that the concept of GPs does not require the sets to be compact translates of one another). After slight moves of  $B$  and  $C$  the sets  $A$ ,  $B$ ,  $C$ , and  $X$  are disjoint and the two lines mentioned induce the desired GPs and have small slopes.

When  $K^+$  is not very long,  $X \cap B$  and  $A \cap C$  are non-empty (but at most 1-dimensional) so the procedure for construction of the desired GPs has to be modified slightly. Now  $K^+$  is assumed to have at least one smooth endpoint. As the Type of any  $K$ -family is clearly invariant under an affine transformation of the plane, we can assume that the right endpoint of  $K^+$  (and thus the left endpoint of  $K^-$ ) is smooth. We start by moving  $A$  and  $X$  equal small distances,  $A$  to the left,  $X$  to the right. Then we get  $A \cap X = B \cap X = A \cap C = \emptyset$  and  $\dim(B \cap A) = \dim(X \cap C) = 1$ . If  $X$  and  $C$  are lifted a little, the descending separating common tangent to  $A$  and  $X$  will induce the GP  $(\text{int}(B) X A \text{ int}(C))$  (here we use the smoothness condition), and the line through the midpoints of  $B \cap A$  and  $X \cap C$  will induce  $(\text{int}(A) \text{ int}(B) \text{ int}(C) \text{ int}(X))$  as before. After slight moves of  $B$  and  $C$  we again have disjointness and the desired GPs.

It is easily checked in both cases that the four sets have no other GP, and it is then clear how to construct, in both cases, a 4-good sequence for  $K$ .  $\square$

## 8 Type 5

Here we have the following theorem.

**Theorem 6.** *Let the oval  $K$  be centrally symmetric.*

- a)  $K$  admits Type 5<sub>3</sub> if and only if it is not a parallelogram.
- b)  $K$  is 5-good if and only if it has a long side with at least one smooth endpoint.

*Proof of Theorem 6.* Here a) follows from Theorem 2b) and c). For b) we start with the necessity and assume  $K$  to be 5-good for a given direction. Let  $K^+$  (defined as in section 7) have both endpoints non-smooth, and let  $A, B, C$  be the limit translates obtained as in the proof of Theorem 4, although  $B$  here plays the role of  $A$  there, so in  $F_n$  we have  $A_n, B,$  and  $C_n$ . Now Lemma 1 requires the slope of the separating common tangents for  $B$  and  $A_n, B$  and  $C_n$  to converge to zero. This requires (remember the central symmetry) both  $A_n$  and  $C_n$  to be above the  $x$ -axis for all large  $n$ , but then the GP  $A_nBC_n$  does not exist.

For the sufficiency, let  $K$  have a long side with a smooth endpoint. During the proof of Theorem 4 we noted in the case when  $K^+$  is long, but not very long, that  $K$  is then, because of the smooth endpoint, 5-good. The argument given there also works when  $K^+$  is very long (in that case we gave a simpler argument for just 3-goodness).  $\square$

## 9 Type 6

This Type is interesting as a  $K$  which admits Type 6 can easily be seen to admit the other Types. Our theorem about 6-goodness is

**Theorem 7.** *Let the oval  $K$  be centrally symmetric.*

- a) If  $K$  is 6-good, it has a long side with at least one smooth endpoint.
- b) If  $K$  has a very long side with at least one smooth endpoint, or a long side with both endpoints smooth, then  $K$  is 6-good.

*Proof of Theorem 7.* A  $K$ - $n$ -family of Type 6 becomes a  $K$ - $(n-1)$ -family of Type 5 by omission of  $X$ . Therefore a) follows from Theorem 6 b). For the proof of b) it is convenient to interchange  $A$  and  $B$ , and  $C$  and  $X$ , in the definition of Type 6. Thus the proof of b) will consist in finding almost horizontal lines inducing GPs  $(BACX)$ ,  $(ABCX)$ , and  $(BXAC)$  for appropriately chosen translates  $A, B, C,$  and  $X$ .

We start out as in the proof of Theorem 5. In the first case we can assume that the left endpoint of  $K^+$  is smooth. We first translate  $X$  and  $C$  by  $(-\epsilon, \epsilon^2)$  for some positive  $\epsilon$ . If  $\epsilon$  is small enough, the falling, separating, common tangent for  $X$  and  $A$  will induce the GP  $(\text{int } B)XA(\text{int } C)$ . Consider  $t_\epsilon$ , the upper tangent to  $A$  from  $(1 - \epsilon, \epsilon^2)$  ( $= X \cap C$ ). The smoothness condition implies that we may assume (possibly after choosing a smaller  $\epsilon$ ) that  $t_\epsilon \cap A$  is a segment or a smooth point. We may also assume, after a slight move of  $B$ , that  $t_\epsilon \cap B = t_\epsilon \cap A$ . Now smoothness implies that  $t_\epsilon$  induces the GP  $A(\text{int } C)(\text{int } X)$ . This means that if we move  $B$  slightly away from  $A$  it is possible, by wriggling  $t_\epsilon$  a little, to get GPs  $AB(\text{int } C)(\text{int } X)$  and  $BA(\text{int } C)(\text{int } X)$ . Thus we have the desired GPs after moving  $X$  a little away from  $C$ .

If  $K^+$  is long, but not very long, we start by translating  $X$  and  $C$  by  $(\epsilon, 0)$ , where  $\epsilon > 0$ . This is to free  $C$  from  $A$ , and  $X$  from  $B$  and  $A$ . Now translate  $X$  and  $C$  by  $(0, \epsilon^2)$ . If  $\epsilon$  is small enough the falling, separating, common tangent for  $X$  and  $A$  will (now by the smoothness condition) induce  $(\text{int}(B) XA \text{int}(C))$ . Just as above we may assume  $t_\epsilon \cap A$  to be a smooth point or a segment, while  $t_\epsilon \cap A = t_\epsilon \cap B$ , where  $t_\epsilon$  is the upper tangent to  $A$  from  $(1 + \epsilon, \epsilon^2)$ . Now  $X$  and  $C$  are moved slightly against each other so that  $X \cap C$  becomes a segment met by  $t_\epsilon$  in an interior point, and then one can again move  $X$  away from  $C$ ,  $A$  away from  $B$ , and wriggle  $t_\epsilon$  to get the desired GPs.  $\square$

As a corollary to *b)* of Theorem 6 and *a)* of Theorem 7 we get a nice result about two important classes of ovals.

**Theorem 8.** *Assume that the oval  $K$  is either strictly convex, or a polygon. Then every sufficiently large  $K$ -family has at most two geometric permutations*

*Proof of Theorem 8.* If  $K$  is strictly convex (i.e.  $K$  has no segment in its boundary) then the same applies to  $K'$ , the Minkowski symmetrization of  $K$ , as will follow from the next section. Thus  $K'$  does not have a long side, and so is neither 5-good nor 6-good. But, as shown in section 5, to every  $K$ -family of Type  $x_n$  there corresponds a  $K'$ -family of Type  $x_n$  (and conversely). Thus  $K$  is neither 5-good nor 6-good, which implies the statement in the theorem.

If  $K$  is a polygon, it is well known that  $K'$  is also a polygon, so that none of its sides has a smooth endpoint. Thus the theorem holds for  $K'$  and hence, as above, for  $K$ .  $\square$

**Remark.** We do not know whether, in Theorem 7*b)*, the smoothness at both ends of  $K^+$  is really necessary, or whether a different kind of condition has to be introduced for the case when  $K^+$  is not very long.

## 10 More on symmetrization

Theorems 4-7 were all formulated for a centrally symmetric  $K$ , for the sake of simplicity. But they have their counterparts for a general  $K$ , which can be found once one is aware of how certain properties of  $K$  relate to the same properties of  $K'$ . Some of these properties were discussed in Section 5, and the remaining ones are dealt with below.

To start with let  $K$  intersect its horizontal tangents  $t^+$  and  $t^-$  in segments of length  $l^+$  and  $l^-$ , one or both of which may be degenerate. The corresponding segments of  $K - K$  are determined by the former and one finds their lengths to be  $l^+ + l^-$ , so that for  $K'$  their lengths are  $(l^+ + l^-)/2$ . In particular if those of  $K'$  are degenerate those of  $K$  are, too. Similar statements hold for all directions.

As was observed in section 6, in a centrally symmetric oval a central chord is at least as long as any parallel chord. Applying this to  $K'$ , and recalling the construction of  $K'$  we find that for any given direction  $K$  and  $K'$  have the same length of maximal chords in that direction. This fact makes it easy to find the counterparts of those theorems where long and very long segments enter, once we have realized certain connections between smoothness of  $K$  and smoothness of  $K'$ .

Consider a  $K$  for which the upper horizontal tangent  $t^+$  intersects  $K$  in the segment  $[p^+, q^+]$  with  $p^+$  to the left, and assume that  $K$  is smooth at  $p^+$ . Then we know that the horizontal direction is 2-good for  $K$ , and that  $t^+$  is the limit of tangents with positive slopes having 2-good directions for  $K$ . From section 5 we then know that the horizontal direction is 2-good for  $K'$  and a limit for 2-good ascending directions for  $K'$ . From central symmetry it now follows that the left endpoint of the top segment of  $K'$  is smooth, and so is its reflection in the center of  $K'$ , which is of course the right endpoint of the base segment. Thus smoothness transfers from  $K$  to  $K'$ .

The reasoning above can be reversed. It will then show that if  $K'$  has a smooth endpoint at, say, the left end of its top boundary segment, then  $K$  will have either a top boundary segment with smooth left endpoint, a base boundary segment with smooth right endpoint, or (if not any of these) a horizontal tangent which intersects  $K$  in one point and is the limit of ascending tangents to  $K$ .

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## About Authors

Andrei Asinowski and Meir Katchalski are at the Faculty of Mathematics, Technion – Israel Institute of Technology, Haifa 3200, Israel.

Andreas Holmsen and Helge Tverberg are at the Departement of Mathematics, University of Bergen, Johs. Brunsgt. 12, 5008 Bergen, Norway.

Email addresses:

`andrei@techunix.technion.ac.il` (A. Asinowski),

`andreash@mi.uib.no` (A. Holmsen),

`meirk@techunix.technion.ac.il` (M. Katchalski),

`tverberg@mi.uib.no` (H. Tverberg).