DYCK PATHS WITH COLOURED ASCENTS

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Abstract

We introduce a notion of Dyck paths with coloured ascents. For several ways of colouring, when the set of colours is itself some class of lattice paths, we establish bijections between sets of such paths and other combinatorial structures, such as non-crossing trees, dissections of a convex polygon, etc. In some cases enumeration gives new expression for sequences enumerating these structures.

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1. Introduction

1.1. **Coloured Dyck paths.** A *Dyck path of length* $2n$ is a sequence P of letters U and D, such that $\#(U) = \#(D) = n$ (where # means "number of") in P, and $\#(U) \geq \#(D)$ in any initial subsequence of P. A Dyck path of length $2n$ is usually represented graphically as a lattice path from the point $(0, 0)$ to the point $(2n, 0)$ that does not pass below the x-axis, where U is the upstep $(1, 1)$ and D is the downstep $(1, -1)$. The set of all Dyck paths of length $2n$ will be denoted by $\mathcal{D}(n)$. We shall also denote $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$. It is well known that $|\mathcal{D}(n)|$ is equal to the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (see [11, Page 222, Exercise 19(i)]).

We remind some standard notation on Dyck paths. A maximal sequence of k consecutive U's (that is, not preceded or followed immediately by another U) in a Dyck path will be called a k*-ascent*. Any sequence of k consecutive U's (resp. D's) will be denoted by U^k (resp. by D^k). The Dyck path U^kD^k will be called a *pyramid of length* $2k$ and denoted by Λ^k. For any U (resp. D), its *matching downstep* (resp. *matching upstep*) is the closest, from right (resp. from left), occurrence of D (resp. of U) such that the number of U's and the number of D's between them are equal. Graphically, U and D match each other if D is right to U, they appear at the same level, and no segment of the path appears at the same level between them. A subsequence DU is called a *valley*.

In this paper we present the following generalization of Dyck paths. Let $\mathfrak{L} = \{\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \dots\}$ be a sequence of sets, and let $a_k = |\mathcal{L}_k|$. We colour all ascents in a Dyck path, when the set of colours for each k-ascent is $\mathcal{L}_k$. In this way we obtain *Dyck paths with ascents coloured by* $\mathfrak{L}$ (shortly *Dyck paths coloured by* $\mathfrak{L}$, or *coloured Dyck paths*). Each Dyck path P produces thus $\prod a_i$ coloured Dyck paths, when the product is taken over the lengths of all ascents in P. Coloured Dyck paths will be denoted by capital letters with "hat", e. g. $\hat{P}$. The pyramid Λ^k with U^k coloured by a specified colour $C \in \mathcal{L}(k)$ will be denoted by $\Lambda^k\langle C\rangle$. The set of all Dyck paths of length $2n$ coloured by members of $\mathfrak{L}$ will be denoted by $\mathcal{D}^{\mathfrak{L}}(n)$. We also denote $\mathcal{D}^{\mathfrak{L}} = \bigcup_{n\geq 0} \mathcal{D}^{\mathfrak{L}}(n)$.

In order to obtain a general expression enumerating $|\mathcal{D}^{\mathfrak{L}}(n)|$, we note that any Dyck path can be presented uniquely in the form

$$U^k DP_k DP_{k-1} \dots DP_1,$$

where $P_k, P_{k-1}, \dots, P_1$ are (possibly empty) Dyck paths. Therefore $M(x)$, the generating function for the sequence $\{|\mathcal{D}^{\mathfrak{L}}(n)|\}_{n\geq 0}$, satisfies

$$M(x) = a_0 + a_1 x M(x) + a_2 x^2 M^2(x) + \dots,$$

or

$$M = A(xM), \tag{1}$$

where $A(x) = \sum_{i\geq 0} a_i x^i$ is the generating function for the sequence $\{|\mathcal{L}_i|\}_{i\geq 0}$.

We choose $\mathfrak{L}$ to be certain classes of lattice paths, and we find that in these cases $M(x)$ turns out to be the generating function for sequences enumerating familiar classes of combinatorial objects. We study this phenomenon by means of bijections between $\mathcal{D}^{\mathfrak{L}}(n)$ and these classes. For example, in Section 2 we consider $\mathcal{L}_k = \mathcal{D}(k)$, and thus $a_k = C_k = \frac{1}{k+1}\binom{2k}{k}$. We shall show the bijection between Dyck paths coloured in this way and non-crossing trees (to be defined in Section 1.2).

1.2. **Non-crossing trees.**

A *non-crossing tree (an NC-tree) on* $[n]$ is an ordered tree which can be represented by a drawing in which the vertices are points on a circle, labeled by $\{1, 2, \dots, n\}$ clockwise, and the edges are non-crossing straight segments. The vertex 1 will be also called *the root*, and we shall depict it as a top vertex. Non-crossing trees have been studied by Chen et al. [3], Deutsch et al. [4, 5], Flajolet et al. [6], Hough [7], Noy et al. [8], and Panholzer et al. [10]. Denote the set of all NC-trees on $[n]$ by $\mathcal{NC}(n)$. It is well known that $|\mathcal{NC}(n+1)| = \frac{1}{2n+1}\binom{3n}{n}$.

Consider the NC-trees on $[n]$ with the property: The vertices on the path from the root 1 to any other vertex appear in increasing order. Such trees are called therefore *non-crossing increasing trees* ("NCI-trees"); the set of NCI-trees on $[n]$ will be denoted by $\mathcal{NCI}(n)$. We also denote $\mathcal{NC} = \bigcup_{n\geq 0} \mathcal{NC}(n+1)$ and $\mathcal{NCI} = \bigcup_{n\geq 0} \mathcal{NCI}(n+1)$.

1.3. **The results.**

We establish bijections between $\mathcal{D}^{\mathfrak{L}}(n)$ and other combinatorial structures, for some specific choices of $\mathfrak{L}$. The main result is the following theorem:

Theorem 1. *There is a bijection between the set of Dyck paths of length* $2n$ *with each* k*-ascent coloured by a Dyck path of length* $2k$ *and the set of non-crossing trees on* $[n+1]$.

Other results are special cases and variations of this theorem.

2. Dyck paths coloured by Dyck paths

Let $\mathfrak{D} = \{\mathcal{D}(0), \mathcal{D}(1), \mathcal{D}(2), \dots\}$. In this Section we consider $\mathcal{D}^{\mathfrak{D}}(n)$, the set of Dyck paths of length $2n$ with k-ascents coloured by Dyck paths of length $2k$, i. e. we take $\mathcal{L}_k = \mathcal{D}(k)$.

First we introduce a convenient way to depict Dyck paths coloured in this way. Given a k-ascent of a path $\hat{P}$ coloured by a path C of length $2k$, we draw a copy of C, rotated by 45° and scaled by $1/\sqrt{2}$, between the endpoints of the ascent. We refer to P as the *base path*, and to C as an *attached path*. For a downstep Y of an attached path, its *matching base downstep* is the downstep of the base path matching the upstep of the base path which is situated below Y. Figure 1 presents in this way the Dyck path $U^5D^2U^3D^6$ with U^5 coloured by $UUDUUUDDDUD$ and U^3 coloured by $UUDUDD$, and illustrates the notion of the matching base downstep.

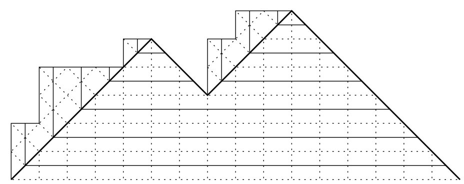

Figure 1. A Dyck path with k-ascents coloured by Dyck paths of length $2k$.

2.1. **Enumeration.** We enumerate $\mathcal{D}^{\mathfrak{D}}(n)$. The generating function for $\{|\mathcal{D}(n)|\}_{n\geq 0}$ is

$$\frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + \dots.$$

Substituting this in (1), we obtain

$$M = \frac{1-\sqrt{1-4xM}}{2xM}.$$

After simplifications, we have $M - 1 = xM^3$. Denoting $L = M - 1$ and applying Lagrange's inversion formula (see [11, Section 5.4] and [14, Section 5.1]) on $L = x(L+1)^3$, we get that the coefficient of x^n in L is

$$[x^n]L = \frac{1}{n}[L^{n-1}](L+1)^{3n} = \frac{1}{n}\binom{3n}{n-1} = \frac{1}{2n+1}\binom{3n}{n}.$$

Thus we have $|\mathcal{D}^{\mathfrak{D}}(n)| = |\mathcal{NC}(n+1)|$.

We shall construct, for each $n \geq 0$, a bijective function $\varphi_n : \mathcal{D}^{\mathfrak{D}}(n) \to \mathcal{NC}(n+1)$. It will be presented as a restriction of a bijective function $\varphi : \mathcal{D}^{\mathfrak{D}} \to \mathcal{NC}$. The function φ will be constructed by the following steps: In Subsection 2.2 we describe a recursive procedure of decomposing a Dyck path into pyramids. In Subsection 2.3 we construct a bijection $\vartheta : \mathcal{D} \to \mathcal{NCI}$. In Subsection 2.4 we first define φ for coloured pyramids and then, using observations from Subsection 2.2, for all the members of $\mathcal{D}^{\mathfrak{D}}$. All by all, this will give the function φ, and we shall also show that it is bijective.

2.2. **Decomposition of a Dyck path into pyramids.** Let P be a Dyck path. Recall that it can be presented uniquely in the form

$$P = U^k D P_k D P_{k-1} \dots D P_1, \tag{2}$$

where $P_k, P_{k-1}, \dots, P_1$ are (possibly empty) Dyck paths. We say that the first k upsteps and the matching downsteps constitute *the base pyramid of* P and $P_k, P_{k-1}, \dots, P_1$ are *appended* to it, and denote this by $P = \Lambda^k * [P_k, P_{k-1}, \dots, P_1]$. Then we decompose in the same way all nonempty paths among $P_k, P_{k-1}, \dots, P_1$, and continue the process recursively. Since $P_k, P_{k-1}, \dots, P_1$ are shorter than P, the process is finite, and it stops when all the paths participating in the decomposition are empty. Thus we obtain *the complete decomposition of* P.

The complete decomposition of a Dyck path can be represented by an ordered tree: Given $P = \Lambda^k * [P_k, P_{k-1}, \dots, P_1]$, we represent it by Λ^k as the root with children $P_k, P_{k-1}, \dots, P_1$. Then we do the same for $P_k, P_{k-1}, \dots, P_1$ and continue recursively, until all the leaves are empty paths. We denote the obtained ordered tree by $OT(P)$. A Dyck path P is easily restored from $OT(P)$.

An example of complete decomposition is

$$\begin{aligned}
&U^4DU^2DUDDDU^2DDDDUDU^2DUD = \\
&\quad = \Lambda^4 * [U^2DUDD, U^2DD, \emptyset, UDU^2DDUD] = \\
&\quad = \Lambda^4 * [\Lambda^2 * [UD, \emptyset], \Lambda^2 * [\emptyset, \emptyset], \emptyset, \Lambda^1 * [U^2DDUD]] = \\
&\quad = \Lambda^4 * [\Lambda^2 * [\Lambda^1 * [\emptyset], \emptyset], \Lambda^2 * [\emptyset, \emptyset], \emptyset, \Lambda^1 * [\Lambda^2 * [\emptyset, UD]]] = \\
&\quad = \Lambda^4 * [\Lambda^2 * [\Lambda^1 * [\emptyset], \emptyset], \Lambda^2 * [\emptyset, \emptyset], \emptyset, \Lambda^1 * [\Lambda^2 * [\emptyset, \Lambda^1 * [\emptyset]]]].
\end{aligned}$$

Each ascent U^k in a Dyck path corresponds to a Λ^k in the complete decomposition. Moreover, each upstep U corresponds to an edge in the ordered tree: the edge from a pyramid Λ^k to its i'th, right-to-left, son (that is, to the base pyramid of P_i) corresponds to the i'th upstep of Λ^k. In one of the constructions below we shall use the labeling of upsteps of P corresponding to the right-to-left preorder labeling *of the edges* of $OT(P)$. Besides, the notion of *depth* will be used: the base pyramid of P has the depth 0; and the base pyramid of a Dyck path appended to a pyramid of depth i has the depth $i+1$. The notion of depth becomes transparent on the tree. Figure 2 illustrates these notions for the example above.

Note that the decomposition (2) is valid also for coloured Dyck paths: $\hat{P} = \hat{U}^k D \hat{P}_k D \hat{P}_{k-1} \dots D \hat{P}_1$, and in the complete decomposition of a coloured Dyck path, each U^k coloured by C results in Λ^k coloured by C. It is also clear how to restore the coloured Dyck path from its complete decomposition to coloured pyramids. The expression (1) enumerates thus also the following structure: ordered trees with n edges, each vertex v coloured by one of $a_{d(v)}$ colours, where $d(v)$ is the outdegree of the vertex. For instance, if each vertex v is coloured by one of $C_{d(v)}$ colours, there are $\frac{1}{2n+1}\binom{3n}{n}$ such trees.

2.3. **A bijection between Dyck paths and NCI-trees.** We begin with a simple bijection $\vartheta : \mathcal{D} \to \mathcal{NCI}$. Let $P \in \mathcal{D}(n)$. Label the downsteps of P by $2, 3, \dots, n+1$ from left to right, and add an auxiliary downstep labeled 1 before the first upstep of P. For each ascent in P, connect the label of the downstep which appears just before the ascent to the labels of all the downsteps matching the upsteps of the ascent. It is easy to see that in this way a member of $\mathcal{NCI}(n+1)$ is obtained. See Figure 3.

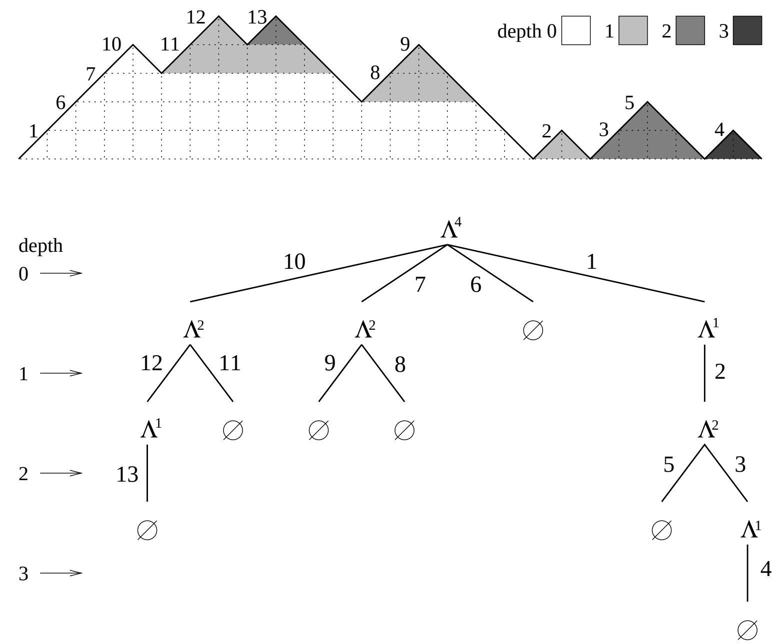

FIGURE 2. The Dyck path $U^4DU^2DUDDDU^2DDDDUDU^2DUD$: the ordered tree of complete decomposition, the ordering of up-steps corresponding to the right-to-left preorder in the tree, the depths.

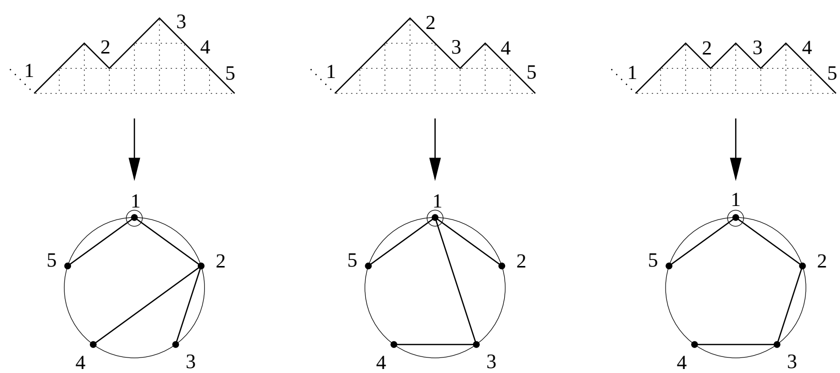

FIGURE 3. The function $\vartheta : \mathcal{D} \to \mathcal{NCI}$.

The function ϑ is invertible: Given $T \in \mathcal{NCI}$, scan it from the root 1 clockwise. Visiting a vertex, add U for each out-edge, then add D for each in-edge (we consider the edges to be oriented so that the root is on the side of the tail of each edge), then move to the next vertex. It is easy to see that the Dyck path P obtained in this way satisfies $\vartheta(P) = T$.

2.4. **Recursive definition of $\varphi : \mathcal{D}^{\mathfrak{D}} \to \mathcal{NC}$.** First we define φ for coloured pyramids. Consider $\Lambda^k\langle C\rangle$ where $C \in \mathcal{D}$. We define $\varphi(\Lambda^k\langle C\rangle) = \vartheta(C)$.

Now we define φ for all coloured Dyck paths. Let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \dots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{D}}$. Suppose that we know $\varphi(\hat{\Lambda}^k)$ and $\varphi(\hat{P}_i)$ for $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, insert a copy of $\varphi(\hat{P}_i)$ into $\varphi(\hat{\Lambda}^k)$ so that the root 1 of $\varphi(\hat{P}_i)$ is mapped to the vertex $i+1$ of $\varphi(\hat{\Lambda}^k)$, and the vertices $2, 3, \dots$ of $\varphi(\hat{P}_i)$ are mapped clockwise to new vertices between i and $i+1$ in $\varphi(\hat{\Lambda}^k)$ (if $\hat{P}_i = \emptyset$ nothing happens). See Figure 4 for the illustration; recall that since the trees obtained at intermediate steps are inserted at appropriate places, their labels change in the process; the bold edges are those corresponding to the base pyramid of the current step.

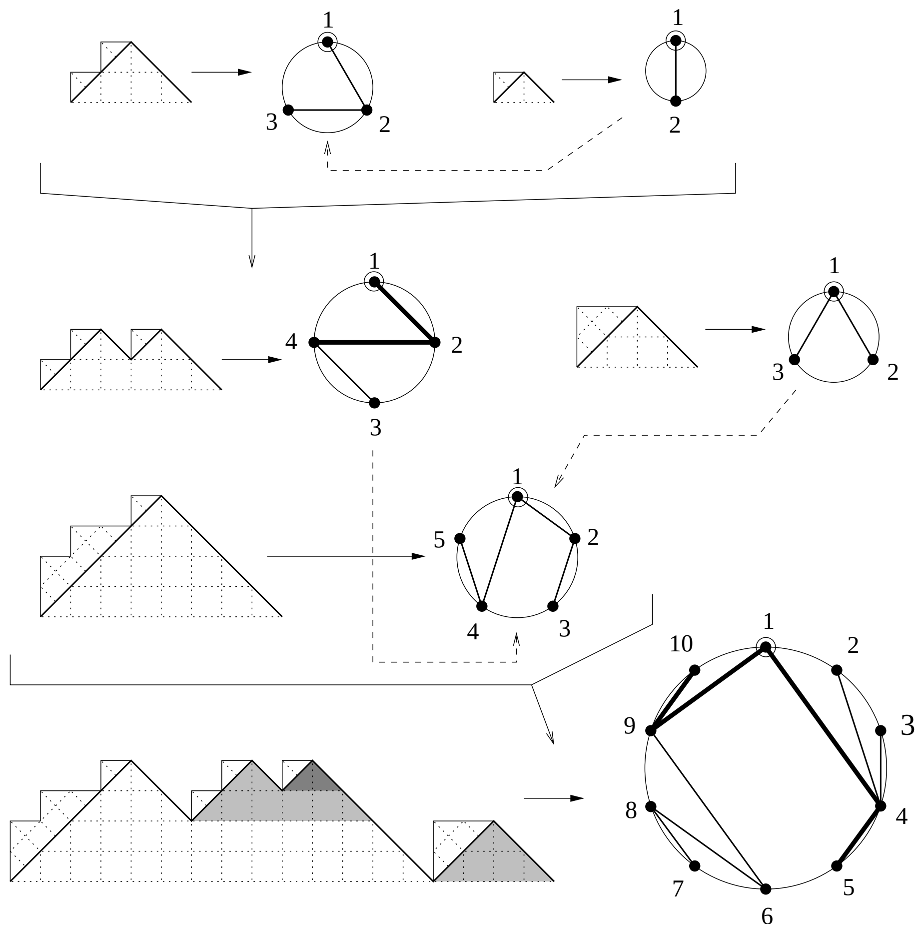

FIGURE 4. The function $\varphi : \mathcal{D}^{\mathfrak{D}} \to \mathcal{NC}$.

The function φ is invertible. Given $T \in \mathcal{NC}$, we want to find $\hat{P} \in \mathcal{D}^{\mathfrak{D}}$ such that $\varphi(\hat{P}) = T$. Take the maximal increasing subtree of T with root 1. After appropriate relabeling of vertices, it forms an NCI-tree V that corresponds to the base pyramid $\hat{\Lambda}^k$ of $\hat{P}$ (k is the number of edges in V) coloured by $\vartheta^{-1}(V)$. For $i = 1, 2, 3, \dots,$

the subtree of T attached at the vertex $i+1$ of V (with this vertex being the root) corresponds to $\hat{P}_i$ which is determined recursively. This allows to restore $\hat{P}$.

It is clear that if $\hat{P} \in \mathcal{D}^{\mathfrak{D}}(n)$ then $\varphi(\hat{P}) \in \mathcal{NC}(n+1)$. Thus we have a family of bijections $\varphi_n : \mathcal{D}^{\mathfrak{D}}(n) \to \mathcal{NC}(n+1)$, for all $n \geq 0$.

This completes the proof of Theorem 1.

2.5. **An explicit description of φ.** In the recursive definition of φ, the correspondence between attaching Dyck paths to a base path and attaching branches in NC-trees is transparent, as well the fact that φ is invertible. However, φ has also the following explicit description. Let $\hat{P} \in \mathcal{D}^{\mathcal{D}}(n)$. Represent it as a base Dyck path with a Dyck path attached to each ascent (as on Figure 1). Label the downsteps *of the base path* by $2, 3, \ldots, n+1$ from right to left, and add an auxiliary downstep labeled by 1 before the first upstep of the path. Label each downstep D *of an attached path* by the label of its matching base downstep. Then construct an NC-tree as follows: For each ascent *of an attached path*, connect the label of the downstep which appears just before the ascent (this downstep may be either on the base or on an attached path) to all the downsteps matching the upsteps of the ascent. See Figure 5 for an example: for the given coloured Dyck path, we get the NC-tree with the edges $1 \to 4, 9;\ 4 \to 2, 3, 5;\ 9 \to 6, 10;\ 6 \to 8;\ 8 \to 7$ (compare with Figure 4).

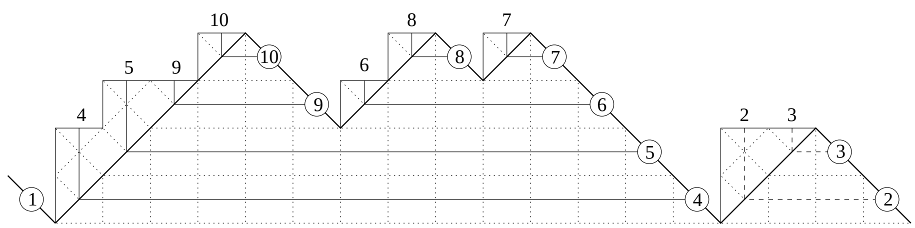

FIGURE 5. The explicit description of φ.

Let a, b, c be three vertices of $\varphi(\hat{P})$, and suppose that $a \to b$ and $b \to c$ are edges of this tree. If the labels a, b, c appear in the clockwise (resp. counterclockwise) order, we say that there is a *clockwise* (resp. *counterclockwise*) *turn* at b. In the bijection φ, a valley *in the base path* of $\hat{P}$ corresponds to a vertex of $\varphi(\hat{P})$ with a counterclockwise turn; a valley *in an attached path* of $\hat{P}$ corresponds to a vertex of $\varphi(\hat{P})$ with a clockwise turn; in both cases the label of the vertex is equal to the label of the downstep forming the valley (note that two types of turns may occur at one point: this happens at points 4 and 9 in our example).

3. DYCK PATHS COLOURED BY DYCK PATHS WITH ASCENTS OF BOUNDED LENGTH

In this section we consider in the role of colours Dyck paths with ascents of bounded length.

Let $\mathcal{M}^m(n)$ be the set of Dyck paths of length $2n$ which avoid U^{m+1}, i. e. with ascents of length $\leq m$. Consider the sequence $\{|\mathcal{M}^m(n)|\}_{n\geq 0}$, for fixed natural m. For $m = 1$ it is the all-1 constant sequence. For $m = 2$ it is the sequence of Motzkin numbers [9, A001006]. The sequences for $m = 3$ and 4 are [9, A036765, A036766] (respectively). For $m \geq n$ we have $|\mathcal{M}^{\geq n}(n)| = C_n$. In this sense, the sequence $\{|\mathcal{M}^m(n)|\}_{n\geq 0}$ may be considered as a generalization of the sequence of Motzkin numbers, and the sequence of sequences $\{\{|\mathcal{M}^m(n)|\}_{n\geq 0}\}_{m\geq 1}$ "converges", with $m \to \infty$, to the sequence of Catalan numbers. Among the structures enumerated by $|\mathcal{M}^m(n)|$, there are

- The set of ordered trees on $n+1$ vertices, all of out-degree $\leq m$.
- The set of all partitions of the vertices of a convex labeled n-polygon into blocks of size $\leq m$ with disjoint convex hulls.

It should be noted that several different kinds of generalized Motzkin numbers can be found in the literature; see [1] for another sequence of sequences which lie between Motzkin and Catalan numbers.

Denote $\mathfrak{M}^m = \{\mathcal{M}^m(0), \mathcal{M}^m(1), \mathcal{M}^m(2), \dots\}$. We consider $\mathcal{D}^{\mathfrak{M}^m}(n)$ for fixed m, and prove the following:

Theorem 2. *There is a bijection between $\mathcal{D}^{\mathfrak{M}^m}(n)$ and the set of partitions of the vertices of a labeled convex $(2n)$-polygon into blocks of even size $\leq 2m$ with disjoint convex hulls. The cardinality of both sets is $\sum_{p=0}^{(n-1)/m} \frac{(-1)^p}{n-mp}\binom{n-mp}{p}\binom{3n-mp-p}{n-mp-1}$.*

We shall use the following notation for shortness. A partition of a convex polygon with n labeled vertices into blocks with disjoint convex hulls will be called a *non-crossing partition (NC-partition) of* $[n]$. A partition of a convex polygon with $2n$ labeled vertices into blocks of even size $\leq 2m$ with disjoint convex hulls (as in Theorem 2) will be called an *even-($\leq 2m$)-NC-partition of* $[2n]$.

3.1. **Enumeration.** The generating function $A(x)$ for $\{|\mathcal{M}^m(n)|\}_{n\geq 0}$ satisfies

$$A(x) = 1 + xA(x) + x^2A^2(x) + \cdots + x^mA^m(x).$$

Substituting this in (1), we obtain that the generating function $h_m(x)$ for $\{|\mathcal{D}^{\mathfrak{M}^m}(n)|\}_{n\geq 0}$ satisfies

$$h_m(x) = 1 + xh_m^2(x) + x^2h_m^4(x) + \cdots + x^mh_m^{2m}(x),$$

which is equivalent to

$$h_m(x) - 1 = xh_m^3(x) - (xh_m^2(x))^{m+1}.$$

Applying the Lagrange inversion formula on

$$h_m(x,a) - 1 = a\left(xh_m^3(x,a) - (xh_m^2(x,a))^{m+1}\right),$$

we obtain

$$h_m(x,a) - 1 = \sum_{\ell\geq 1} \frac{a^\ell x^\ell}{\ell} \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell}(-1)^j x^{mj}\binom{3\ell}{i}\binom{\ell}{j}\binom{(2m-1)j}{\ell-1-i},$$

which implies that the coefficient of x^n in $h_m(x) = h_m(x,1)$ is

$$[x^n](h_m(x)) = \sum_{p=0}^{(n-1)/m} \frac{(-1)^p}{n-mp}\binom{n-mp}{p}\sum_{i=0}^{n-mp-1}\binom{3n-3mp}{i}\binom{2mp-p}{n-mp-1-i} =$$

$$= \sum_{p=0}^{(n-1)/m} \frac{(-1)^p}{n-mp}\binom{n-mp}{p}\binom{3n-mp-p}{n-mp-1}. \tag{3}$$

This is the cardinality of $\mathcal{D}^{\mathfrak{M}^m}(n)$.

3.2. **Even non-crossing partitions of** $[2n]$**.** Consider a convex polygon with $2n$ vertices labeled by $\{1, 2, \ldots, 2n\}$, 1 being the root (the top point). Denote by $\mathcal{E}^m(n)$ the set of all even-($\leq 2m$)-NC-partitions of $[2n]$. Denote also $\mathcal{E}(n) = \bigcup_{m\geq 1} \mathcal{E}^m(n)$, $\mathcal{E} = \bigcup_{n\geq 0, m\geq 1} \mathcal{E}^m(n)$.

For all $n \geq 0$, $m \geq 1$ we shall construct a bijection $\rho_{n,m} : \mathcal{D}^{\mathfrak{M}^m}(n) \to \mathcal{E}^m(n)$. It will be presented as a restriction of a bijection $\rho : \mathcal{D}^{\mathfrak{D}} \to \mathcal{E}$.

We start with a bijection ψ from $\mathcal{D}$ to $\bar{\mathcal{E}}$, the set of all NC-partitions of $[n]$.

Given $P \in \mathcal{D}(n)$ we construct the partition $\psi(P) \in \bar{\mathcal{E}}(n)$ according to the following rule. Label the downsteps of P by $1, 2, \ldots, n$ from left to right. Each ascent in P contributes to $\psi(P)$ the block consisting of the labels of D's which match the U's in the ascent, see Figure 6. It is easy to see that ψ is well defined and invertible.

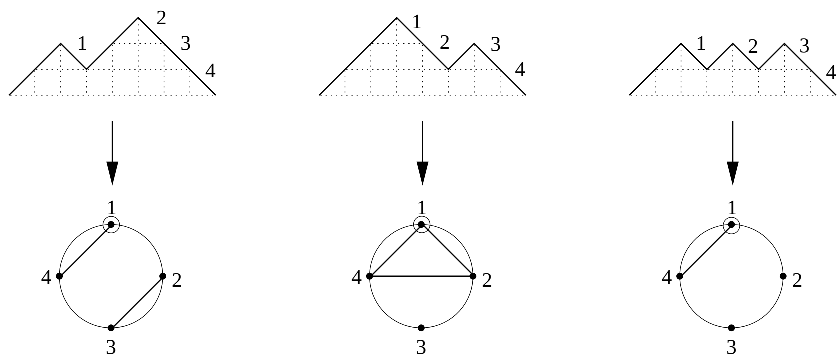

FIGURE 6. The function $\psi : \mathcal{D} \to \bar{\mathcal{E}}$.

Now we define $\rho : \mathcal{D}^{\mathfrak{D}} \to \mathcal{E}$. First we define it for coloured pyramids.

Let $M \in \mathcal{D}$. We define $\rho(\Lambda^k\langle M\rangle)$ to be a "duplicated $\psi(M)$", i. e. for each block $\{x_1, x_2, x_3, \ldots\}$ in $\psi(M)$, we have the block $\{2x_1 - 1,\, 2x_1,\, 2x_2 - 1,\, 2x_2,\, 2x_3 - 1,\, 2x_3, \ldots\}$ in $\rho(\Lambda^k\langle M\rangle)$.

Now let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \ldots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{M}^m}(n)$, and suppose we know $\rho(\hat{\Lambda}^k)$ and $\rho(\hat{P}_i)$ for $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$, insert a copy of $\rho(\hat{P}_i)$ into $\rho(\hat{\Lambda}^k)$ so that all the points of $\rho(\hat{P}_i)$ are mapped clockwise to new points between $2i - 1$ and $2i$. The obtained partition is $\rho(\hat{P})$. See Figure 7 for an illustration.

The function ρ is invertible. Given $T \in \mathcal{E}$, we want to find $\hat{P}$ such that $\rho(\hat{P}) = T$. Consider T as the union of polygons and choose points of T, beginning from the root 1 and proceeding as follows: if i is an even point, pass to $i + 1$; if i is an odd point, pass to the next point belonging to the same block. Denote by V the union of polygons formed by the chosen vertices after appropriate relabeling of vertices (these polygons have bold edges in Figure 7). It corresponds to the base pyramid of $\hat{P}$ with colouring determined by joining all pairs of points $\{2i - 1, 2i\}$ into one point i and then applying ψ^{-1}. For $i = 1, 2, 3, \ldots$, the part of T between the points $2i - 1$ and $2i$ of V (with the point next to $2i - 1$ as the root) corresponds to P_i which is determined recursively. This allows to restore $\hat{P}$.

It is easy to see that if $\hat{P} \in \mathcal{D}^{\mathfrak{M}^m}(n)$ then $\rho(\hat{P}) \in \mathcal{E}^m(n)$. In particular, each k-ascent in the base path results in a partition of a polygon with $2k$ vertices, and each k-ascent in an attached path results in a block of size $2k$. Thus we have a family

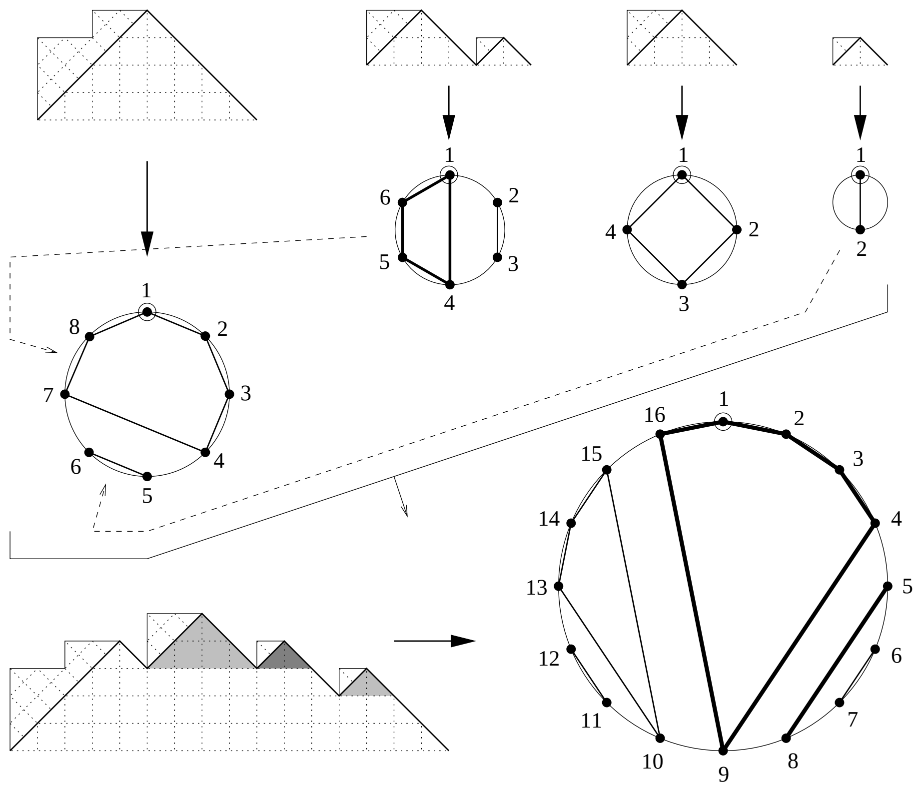

FIGURE 7. The function $\rho : \mathcal{D}^{\mathcal{D}} \to \mathcal{E}$.

of bijections $\rho_{n,m} : \mathcal{D}^{\mathfrak{M}^m}(n) \to \mathcal{E}^m(n)$, for all $n \geq 0, m \geq 1$, and this completes the proof of Theorem 2.

3.3. **An explicit description of ρ.** The function ρ has the following explicit description. Label the base path P of $\hat{P}$ by $\{1, 2, \ldots, 2n\}$ according to the right-to-left prepostorder double labeling of the edges of $OT(P)$: each upstep gets the label obtained on the first visiting of the corresponding edge, its matching downstep gets the label obtained on the second visiting ("on return") of the same edge. Then label the steps of attached paths: a downstep of an attached path gets the label of the match of the base path upstep situated below it; the upsteps get the labels of the base path ascent itself, ordered according to their matches (the match of the first downstep gets the smallest label, and so on). See Figure 8: the coloured Dyck path gives the NC-partition to blocks $\{1, 2, 3, 4, 9, 16\}$, $\{5, 8\}$, $\{10, 13, 14, 15\}$, $\{11, 12\}$, $\{6, 7\}$ (compare with Figure 7).

3.4. **Some special cases.** We consider some special cases.

1. Let $m = 1$. Substituting this in (3), we get

$$\sum_{p=0}^{n-1} \frac{(-1)^p}{n-p} \binom{n-p}{p} \binom{3n-2p}{n-p-1},$$

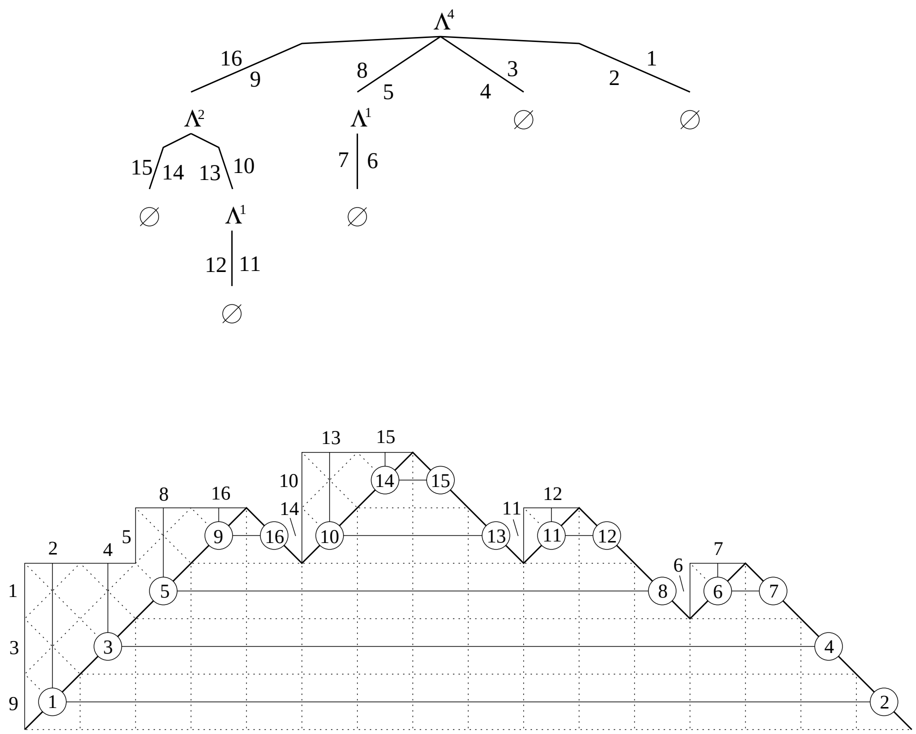

FIGURE 8. The explicit description of ρ.

which is equal to C_n: since $|\mathcal{M}^1(k)| = 1$ for each k, we have $|\mathcal{D}^{\mathfrak{M}^1}(n)| = |\mathcal{D}(n)| = C_n$.

The even NC-partitions of $[2n]$ corresponding to the members of $\mathcal{D}^{\mathfrak{M}^1}(n)$ in the bijection ρ are those in which each block is of the size 2 (all the ways to connect pairs of points of $[2n]$ in convex position by disjoint segments).

2. The case $m = 2$ was studied in [13], where the authors found a formula for $\{|\mathcal{E}^2(n)|\}_{n\geq 0}$ (it is enumerated by [9, A006605]). They also considered the generating function for $\{|\mathcal{E}^m(n)|\}_{n\geq 0}$, and noted that $|\mathcal{E}(n)| = \frac{1}{2n+1}\binom{3n}{n}$.

3. Let $m = n$. Substitute this in (3). The only relevant value of p is 0, and we get therefore

$$\frac{1}{n}\binom{3n}{n-1} = \frac{1}{2n+1}\binom{3n}{n},$$

which is expected: we have $|\mathcal{M}^n(n)| = C_n$ and thus $\mathcal{D}^{\mathfrak{M}^n}(n) = \mathcal{D}^{\mathfrak{D}}(n)$. The corresponding partitions of $[2n]$ are all even NC-partitions.

4. Dyck paths coloured by Fibonacci paths

In this Section we consider a further restriction of Dyck paths taken as colours.

Let $\mathcal{F}^m(n)$ be the set of Dyck paths of length $2n$ which have the form $\Lambda^{k_1}\Lambda^{k_2}\Lambda^{k_3}\ldots$ with $k_i \leq m$ – a concatenation of pyramids of length no more than $2m$. Note that $\mathcal{F}^m(n) \subset \mathcal{M}^m(n)$. It is known that $|\mathcal{F}^2(n)|$ is equal to the $(n+1)$-st Fibonacci number; $|\mathcal{F}^m(n)|$ is the $(n+1)$-st m-generalized Fibonacci number (see [11, A092921]).

The members of $\mathcal{F}^m(n)$ will be therefore called *m-generalized Fibonacci paths*. Besides, denote $\mathcal{F}(n) = \mathcal{F}^n(n)$. We have $|\mathcal{F}(n)| = 2^{n-1}$ for $n > 0$, and $|\mathcal{F}(0)| = 1$.

Denote $\mathfrak{F}^m = \{\mathcal{F}_0^m, \mathcal{F}_1^m, \mathcal{F}_2^m, \dots\}$ and $\mathfrak{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \dots\}$. We consider $\mathcal{D}^{\mathfrak{F}^m}(n)$ for fixed m, and $\mathcal{D}^{\mathfrak{F}}(n)$, and prove the following:

Theorem 3. *There is a bijection between $\mathcal{D}^{\mathfrak{F}^m}(n)$ and the set of diagonal dissections of a labeled convex $(n+2)$-polygon into 3-, 4-, …, $(m+2)$-polygons. The cardinality of both sets is $\sum_{\ell=0}^{n-1} \frac{1}{\ell+1}\binom{n+\ell+1}{\ell} \sum_{i=0}^{(n-1)/m} (-1)^i \binom{n-1-mi}{\ell}\binom{\ell+1}{i}$.*

4.1. **Enumeration.** The generating function of the sequence $\{|\mathcal{F}^m(n)|\}_{n\geq 0}$ is

$$\sum_{n\geq 0} |\mathcal{F}(n)^m| x^n = \frac{1}{1-x-x^2-\cdots-x^m}.$$

Substituting this in (1), we obtain that the generating function $g_m(x)$ for the sequence $\{|\mathcal{D}^{\mathfrak{F}^m}(n)|\}_{n\geq 0}$ satisfies

$$g_m(x) = \frac{1}{1-xg_m(x)-\cdots-(xg_m(x))^m}$$

which is equivalent to

$$g_m(x) - 1 = x(g_m(x))^2 \frac{1-x^m(g_m(x))^m}{1-xg_m(x)}.$$

Applying the Lagrange inversion formula on

$$g_m(x,a) - 1 = ax(g_m(x,a))^2 \frac{1-x^m(g_m(x,a))^m}{1-xg_m(x,a)},$$

we obtain

$$g_m(x,a)-1 = \sum_{\ell\geq 0} \frac{a^{\ell+1}}{\ell+1} \sum_{j\geq 0} \sum_{i=0}^{\ell+1} (-1)^i x^{\ell+j+mi+1} \binom{\ell+j}{j}\binom{\ell+1}{i}\binom{2\ell+2+j+mi}{\ell},$$

which implies that the coefficient of x^n in $g_m(x) = g_m(x,1)$ is

$$\sum_{\ell=0}^{n-1} \frac{1}{\ell+1}\binom{n+\ell+1}{\ell} \sum_{i=0}^{(n-1)/m} (-1)^i \binom{n-1-mi}{\ell}\binom{\ell+1}{i}. \tag{4}$$

This is the cardinality of $\mathcal{D}^{\mathfrak{F}^m}(n)$.

4.2. **Dissections of a convex polygon.** Denote by $\mathcal{R}_m(n)$ the set of all dissections of a labeled convex n-polygon into i-polygons with $i = 3, 4, \dots, m$ by non-crossing diagonals. We shall label the vertices of the n-polygon by $\alpha, 0, 1, \dots, n$ clockwise, the top vertex being α (the root). Denote also $\mathcal{R} = \bigcup_{n\geq 0, m\geq 1} \mathcal{R}_{m+2}(n+2)$.

For all $n \geq 0, m \geq 1$ we construct a bijection $\sigma_{n,m} : \mathcal{D}^{\mathfrak{F}^m}(n) \to \mathcal{R}_{m+2}(n+2)$. It will be presented as a restriction of a bijection $\sigma : \mathcal{D}^{\mathfrak{F}} \to \mathcal{R}$.

Consider $\Lambda^k\langle F\rangle$, a pyramid of length $2k$ coloured by an generalized Fibonacci path $F \in \mathcal{F}(k)$. We define $\sigma(\Lambda^k\langle F\rangle)$ to be the dissection of the convex polygon with $k+2$ vertices, taking a diagonal (α, i) if and only if the path F touches the x-axis at point $(2i, 0)$ (see Figure 9).

Let $\hat{P} = \hat{\Lambda}^k * [\hat{P}_k, \hat{P}_{k-1}, \dots, \hat{P}_1] \in \mathcal{D}^{\mathfrak{F}}$. Suppose that we know dissections $\sigma(\hat{\Lambda}^k)$ and $\sigma(\hat{P}_i)$ for $i = 1, 2, \dots, k$. For each $i = 1, 2, \dots, k$, attach a copy of $\sigma(\hat{P}_i)$ to $\sigma(\hat{\Lambda}^k)$ so that the vertex α of $\sigma(\hat{P}_i)$ is mapped to the vertex $i-1$ of $\sigma(\hat{\Lambda}^k)$, the last vertex of $\sigma(\hat{P}_i)$ is mapped to the vertex i of $\sigma(\hat{\Lambda}^k)$, and the vertices $1, 2, \dots$

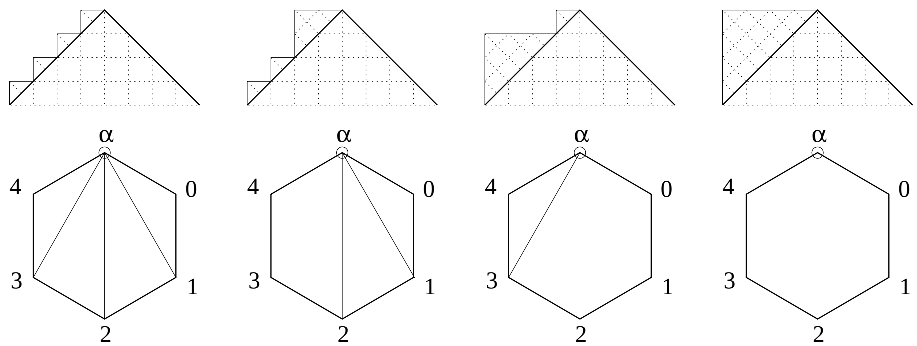

FIGURE 9. Definition of the function σ on pyramids.

of $\sigma(\hat{P}_i)$ are mapped clockwise to new vertices between $i-1$ and i of $\sigma(\hat{\Lambda}^k)$. The side $(i-1,i)$ of $\sigma(\hat{\Lambda}^k)$ becomes thus a diagonal of the new polygon; such a diagonal (along which polygons are pasted to each other) will be called *essential*. After relabeling the vertices we obtain a dissection $\sigma(\hat{P})$. See Figure 10 for an example.

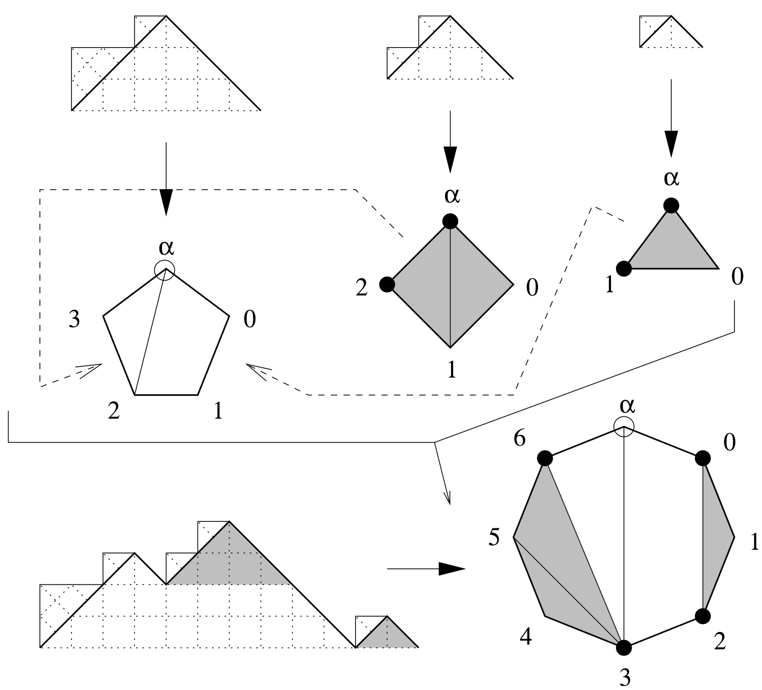

FIGURE 10. The function $\sigma : \mathcal{D}^{\mathfrak{F}} \to \mathcal{R}$.

This function σ invertible: Let $T \in \mathcal{R}$ and we want to find $\hat{P} \in \mathcal{D}^{\mathfrak{F}}$ such that $\sigma(\hat{P}) = T$. Let V be the union of all the polygons in the dissection that have α as a vertex. After the appropriate relabeling of its vertices, V corresponds to the base pyramid of $\hat{P}$ which is restored immediately. The part attached to V along the edge $(i-1,i)$, with $i-1$ as the root, corresponds to $\hat{P}_i$ which is restored recursively.

Each pyramid of length $2j$ in F (a colouring of Λ^k) results in a $(j+2)$-polygon in the dissection of $(k+2)$-polygon. Therefore if $\hat{P} \in \mathcal{D}^{\mathfrak{F}^m}(n)$ then $\sigma(\hat{P}) \in \mathcal{R}_{m+2}(n+2)$. Thus we have a family of bijections $\sigma_{n,m} : \mathcal{D}^{\mathfrak{F}^m}(n) \to \mathcal{R}_{m+2}(n+2)$, for all $n \geq 0, m \geq 1$, and this completes the proof of Theorem 3.

4.3. **An explicit description of σ.** The function σ has the following explicit description. Let $\hat{P} \in \mathcal{D}^{\mathfrak{F}}(n)$. Label the upsteps of the base path by $\alpha, 0, 1, \ldots, n-2$ according to the right-to-left preorder of the edges of the ordered tree representing the complete decomposition of P (see Figure 2). Label the downsteps of the base path by $1, 2, \ldots, n$ from right to left. Then add to each label of a downstep its depth in the complete decomposition of the path. Label the downsteps of the attached path by the label of the match of the upstep of the base path which is situated below this downstep. Now each valley in $\hat{P}$ corresponds to a diagonal in $\sigma(\hat{P})$: Each valley *of the base path* gives the essential diagonal between the vertices labeled as the downstep and the upstep forming the valley; each valley *of an attached path* gives a non-essential diagonal between the label of the first upstep in the ascent to which it is attached and the label of the downstep forming this valley. See Figure 11: this path of the length $12(= 2 \cdot 6)$ gives the dissection of the polygon with 8 vertices labeled $\alpha, 0, 1, \ldots, 6$ by four diagonals: essential $(3,6)$ and $(0,2)$, non-essential $(\alpha, 3)$ and $(3,5)$ (compare with Figure 10).

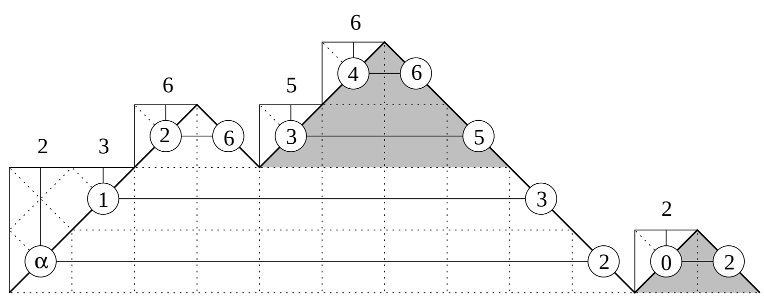

FIGURE 11. The explicit description of σ.

4.4. **Some special cases.** We consider a few special cases.

1. Let $m = 1$. Substitute this in (4). It can be shown that

$$\sum_{i=0}^{n-1} (-1)^i \binom{n-1-i}{\ell} \binom{\ell+1}{i} = 0$$

for $l \in \{0, 1, \ldots n-2\}$. Therefore the whole expression is equal to

$$\frac{1}{n}\binom{2n}{n-1} \sum_{i=0}^{n-1} (-1)^i \binom{n-1-i}{n-1} \binom{n}{i} = \frac{1}{n}\binom{2n}{n-1} = C_n,$$

as expected, since $|\mathcal{F}^1(k)| = 1$ for each k. This agrees with a well-known fact that $|\mathcal{R}_3(n+2)|$, i. e. the number of dissections of $n+2$-polygon into triangles, is C_n (see [11, Page 221, Exercise 19(a)]).

2. Consider the colouring $\mathcal{L}(k) = \{\Lambda^k\}$. In this case we also have $|\mathcal{L}(k)| = 1$ for each k, and therefore $|\mathcal{D}^{\mathfrak{L}}(n)| = C_n$. The corresponding dissections of the labeled polygon with $(n+2)$ vertices are those with the property: No diagonal has α as an endpoint, and each other vertex is an endpoint of at most one diagonal connecting it with a vertex with a greater label. Thus there are C_n such dissections; indeed, they are equivalent to [12, Example (s^4)].

3. Let $m = n$. Substitute this in (4). It is clear that

$$(-1)^i \binom{n-1-ni}{\ell}\binom{\ell+1}{i} = 0$$

for $i > 0$. Therefore the whole expression is equal to

$$\sum_{\ell=0}^{n-1} \frac{1}{\ell+1}\binom{n+\ell+1}{\ell}\binom{n-1}{\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1}\binom{n+\ell+1}{\ell+1}\binom{n-1}{\ell}.$$

This expression defines the sequence of "Little Schröder numbers" [9, A001003]. Indeed, it is well known that it enumerates $\mathcal{R}_n(n+2)$, i. e. all possible dissections of $n+2$-polygon. See [2] for a recent related result.

Let us enumerate $\mathcal{D}^{\mathfrak{F}}(n)$ in another way. The generating function for $\{|\mathcal{F}(n)|\}_{n\geq 0}$ is

$$\sum_{n\geq 0} |\mathcal{F}(n)| x^n = \frac{1-x}{1-2x} = 1 + x + 2x^2 + 4x^3 + \dots.$$

Substituting this in (1), we get

$$M = \frac{1-xM}{1-2xM},$$

which is equivalent to $M - 1 = xM(2M-1)$. Taking $L = M - 1$, we have $L = x(L+1)(2L+1)$, and by Lagrange's inversion formula,

$$[x^n]L = \frac{1}{n}[L^{n-1}]\left((L+1)(2L+1)\right)^n = \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n}{i+1}2^i.$$

This is the cardinality of $\mathcal{D}^{\mathfrak{F}}(n)$ and thus another expression for Little Schröder numbers.

The sequence of Little Schröder numbers enumerates "Little Schröder paths", see [9, A001003]. A *Little Schröder path of length* $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with moves $U = (1,1)$, $D = (1,-1)$, $L = (2,0)$, which does not pass below the x-axis and does not contain an L-step on the x-axis.

Denote the set of all Little Schröder paths of length $2n$ by $\mathcal{LS}(n)$, and $\mathcal{LS} = \bigcup_{n\geq 0} \mathcal{LS}(n)$. According to our result, $|\mathcal{D}^{\mathfrak{F}}(n)| = |\mathcal{LS}(n)|$. We construct a simple direct bijection between these sets:

Observation 4. *There is a bijection between $\mathcal{D}^{\mathfrak{F}}(n)$ and $\mathcal{LS}(n)$. The cardinality of both sets is the n-th Little Schröder number.*

Let $F \in \mathcal{F}(k)$. Represent it by a $\{0,1\}$-sequence $(x_1x_2\dots x_{k-1})$: consider F as a lattice path from $(0,0)$ to $(2k,0)$ and let $x_i = 1$ if F touches the x-axis at the point $(2i,0)$, and $x_i = 0$ otherwise.

Let $\hat{P} \in \mathcal{D}^{\mathfrak{F}}$. Consider the complete decomposition of $\hat{P}$. Each pyramid $\hat{\Lambda}_k$ in this decomposition is coloured by the members of $\{0,1\}^{k-1}$. Replace $\Lambda_k\langle(x_1x_2\dots x_{k-1})\rangle$ with $U^{\beta+1}A_{k-1}A_{k-2}\dots A_2A_1D$ where β is the number of 1's in $(x_1x_2\dots x_{k-1})$,

$A_i = D$ if $x_i = 1$, $A_i = L$ if $x_i = 0$. In this way a Little Schröder path is obtained, see Figure 12 for an example.

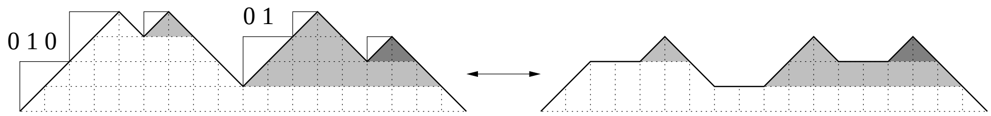

FIGURE 12. The bijection $\mathcal{D}^{\mathfrak{F}}(n) \leftrightarrow \mathcal{LS}(n)$.

This function is easily seen to be invertible: this is based on the fact that any Little Schröder path P may be written in a unique way as $P = U^{\ell} X_k P_k X_{k-1} P_{k-1} \dots X_2 P_2 D P_1$, where each X_i is D or L, and each P_i is a (possibly empty) Little Schröder path.

Besides, the members of $\mathcal{D}^{\mathfrak{F}}(n)$ correspond to the members of $\mathcal{LS}(n)$.

5. Dyck paths coloured by Schröder paths

Finally, we take $\mathcal{L}_k$ to be the set of all Schröder paths of length $2k$. A *Schröder path of length* $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ with moves $U = (1,1)$, $D = (1,-1)$, $L = (2,0)$, which does not pass below the x-axis. The set of all Schröder paths of length $2n$ will be denoted by $\mathcal{S}(n)$. Schröder sequences are enumerated by [9, A006318]. Let $\mathfrak{S} = \{\mathcal{S}(0), \mathcal{S}(1), \mathcal{S}(2), \dots\}$.

Denote by $\mathcal{T}(n)$ the set of all lattice paths from $(0,0)$ to $(3n,0)$ with moves $H = (1,2)$, $G = (2,1)$, and $D = (1,-1)$, that do not pass below the x-axis. It is known that $\mathcal{T}(n)$ is enumerated by [9, A027307].

Observation 5. *There is a bijection between* $\mathcal{D}^{\mathfrak{S}}(n)$ *and* $\mathcal{T}(n)$.

We enumerate $\mathcal{D}^{\mathfrak{S}}(n)$ as follows. The generating function for $\{|\mathcal{S}(n)|\}_{n\geq 0}$ is

$$\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}.$$

Substituting this in (1) we get

$$M = \frac{1 - xM - \sqrt{1 - 6xM + x^2M^2}}{2xM},$$

or, after simplifications, $M - 1 = x(M^3 + M^2)$. Denoting $L = M - 1$ and applying Lagrange inversion formula on $L = x(L+1)^2(L+2)$, we finally get

$$\begin{aligned} [x^n](L) &= \tfrac{1}{n}[L^{n-1}]((L+1)^2(L+2))^n \\ &= \tfrac{1}{n}\sum_{i,j\geq 0}\tbinom{n}{i}\tbinom{n}{j}\tbinom{n}{i+j+1}2^{i+j+1} = \tfrac{1}{n}\sum_{k=0}^{n-1}\tbinom{2n}{k}\tbinom{n}{k+1}2^{k+1}. \end{aligned}$$

This is the cardinality of $\mathcal{D}^{\mathfrak{S}}(n)$, and it is known to be an expression for $|\mathcal{T}(n)|$, see [9, A027307].

We show a simple direct bijection between $\mathcal{D}^{\mathfrak{S}}(n)$ and $\mathcal{T}(n)$.

Consider a member of $\mathcal{D}^{\mathfrak{S}}(n)$. Replace each k-ascent by the Schröder path which colours it, rotated by 45° and scaled by $1/\sqrt{2}$ (as we did earlier, see Figure 1). We obtain a path with steps $U = (1,1)$, $N = (0,1)$, $E = (1,0)$, $D = (1,-1)$.

Replacing $N \to H$, $U \to G$, $E \to D$, $D \to D$, we obtain a member of $\mathcal{T}(n)$. See Figure 13 for an illustration.

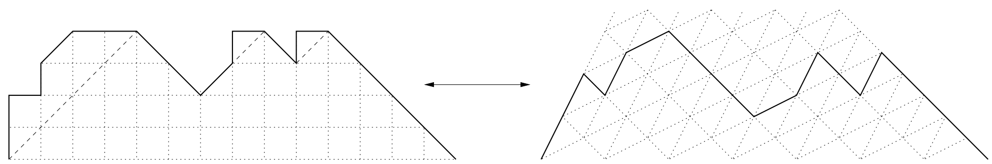

FIGURE 13. The bijection $\mathcal{D}^{\mathfrak{S}}(n) \leftrightarrow \mathcal{T}(n)$.

This correspondence is easily seen to be invertible: Given a member of $\mathcal{T}(n)$, replace $H \to N$, $G \to U$, $D \to E$ or $D \to D$ according to the following rule: if D is the match of an H then $D \to E$, otherwise $D \to D$ (A match of an H is the closest, from right, D such that the number of H's and of D's between them is equal).

A restriction of this correspondence is that between Dyck paths with ascents coloured by Dyck paths and the members of $\mathcal{T}(n)$ with only moves $H = (1, 2)$ and $D = (1, -1)$.

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