

# Lecture Notes on Value Judgements

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**A Note to the Student:** These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

“It is true that convictions can best be supported with experience and clear thinking. On this point one must agree with the extreme rationalist. The weak point of his conception is, however, this, that those convictions which are necessary and determinant for our conduct and judgments, cannot be found solely along this solid scientific way.

“For the scientific method can teach us nothing else beyond how facts are related to, and conditioned by, each other. . . . [Yet] knowledge of what ‘is’ does not open the door directly to what ‘should be’. One can have the clearest and most complete knowledge of what ‘is’, and yet not be able to deduct from that what should be the ‘goal’ of our human aspirations.”  
 Albert Einstein, *Out Of My Later Years*, pp21–22. Thames and Hudson Pub., London, 1950.

## Contents

<b>1</b>	<b>The Problem</b>	<b>2</b>
<b>2</b>	<b>Prior Experience</b>	<b>4</b>
<b>3</b>	<b>Dimensional Normalization</b>	<b>5</b>
<b>4</b>	<b>Analogical Reasoning</b>	<b>9</b>
<b>5</b>	<b>Calibration by Consequence Severity</b>	<b>13</b>
5.1	Robustness function . . . . .	13
5.2	Special Case . . . . .	16
5.3	Calibration by Consequence Severity . . . . .	17

# 1 The Problem

¶ We have learned that, for  $\hat{h}(q, r_c)$ , **bigger is better**.

But now we face the questions:

- How big is big enough?
- What is the utility of a marginal increment in  $\hat{h}$ ?
- How do we evaluate the **numerical value** of  $\hat{h}$   
in terms of

**subjective personal or social values?**

¶ A similar problem arises in probabilistic decision theory.

If  $P_s$  = probability of success,

Then **bigger is better** for  $P_s$  just as for  $\hat{h}$ .

But the same value judgment arises about  $P_s$  as about  $\hat{h}$ :

How big is big enough?

¶ Value judgment as calibration:

How to **calibrate**  $\hat{h}$  in

subjective, value-based linguistic terms

such as: big enough, too small, etc.

¶ We will consider three methods for calibrating  $\hat{h}$ :

1. Prior experience.
2. Dimensional normalization.
3. Severity of Consequences.

All three are in fact cases of **reasoning by analogy**.

**¶ Calibration: incremental or absolute?**

- Incremental calibration: how significant is  $\Delta\hat{h}$ ?
- Absolute calibration: how robust (i.e. safe) is  $\hat{h}$ ?
  
- Incremental calibration: plausible answer with analogical reasoning.
- Absolute calibration:
  - No Solution.
  - Moral version of absolute space: does not have meaning.

## 2 Prior Experience

¶ Suppose we have a system  $S_1$  which:

1. We are familiar with.
2. Performs well.
3. Has robustness  $\hat{h}_1$ .

¶ Now consider a system  $S_2$  which:

- 1'. Is similar to  $S_1$  in important respects.
- 2'. Has unknown performance.
- 3'. Has robustness  $\hat{h}_2$  which is close to  $\hat{h}_1$ .

¶ We may tend to conclude that,

**by analogy** between  $S_1$  and  $S_2$ ,

$S_2$  will tend to perform well, like  $S_1$ .

This suggests that  $\hat{h}_2$  is adequate robustness.

¶ Our reasoning here is:

1.  $S_1$  and  $S_2$  are similar.
2.  $S_1$  performs well.
3. Therefore  $S_2$  will tend to perform well;  
and  $\hat{h}_2$  will tend to be adequate.

This is **reasoning by analogy**:

Things which are **similar in some respects**

**will tend to be similar in other respects** as well.

Reasoning by analogy is **not proof**.

Reasoning by analogy is **plausible inference**.

### 3 Dimensional Normalization

(Section 4.1)

¶ The uncertainty parameter  $h$  of an info-gap model usually has **physical units**.

For instance, if  $u(x)$  = uncertain quantity of pollutant, [kg/m<sup>2</sup>],

whose info-gap model is:

$$\mathcal{U}(h, \tilde{u}) = \{u(x) : |u(x) - \tilde{u}(x)| \leq h\psi(x)\}, \quad h \geq 0 \quad (1)$$

we see that  $[h] = [u(x)] = [\tilde{u}(x)] = [\text{kg/m}^2]$ .

( $\psi(x)$  is a dimensionless envelope.)

Or, for example, if  $u(t)$  = uncertain mechanical stress, [N/m<sup>2</sup>],

whose info-gap model is:

$$\mathcal{U}(h, \tilde{u}) = \left\{ u(t) : \int_0^\infty (u(t) - \tilde{u}(t))^2 dt \leq h^2 \right\}, \quad h \geq 0 \quad (2)$$

then  $[h] = [u]\sqrt{\text{sec}} = \frac{\text{N}\sqrt{\text{s}}}{\text{m}^2}$ .

¶ The robustness  $\hat{h}$  is the least upper bound of a set of  $h$ -values:

$$\hat{h} = \max \{h : \text{Some critical condition holds}\} \quad (3)$$

Thus  $\hat{h}$  and  $h$  always have the same units.

¶ This implies that:

- There is **no inherent calibration** of  $\hat{h}$  in absolute units.
- We cannot say ' $\hat{h} = 10^6$ ' implies ' $\hat{h}$  is large'.
- We must calibrate the units of  $\hat{h}$  in some way.

¶ One type of calibration of  $\hat{h}$  is based on:

- physical reasoning,
- engineering judgment,
- economic judgment,
- etc.

These considerations can **normalize**  $\hat{h}$  to **dimensionless** form.

¶ For instance, consider the envelope-bound info-gap model in eq.(1) on p. 5.

The robustness is dimensional:  $[\hat{h}] = [\tilde{u}] = \text{kg/m}^2$ .

However,  $\frac{\hat{h}}{\tilde{u}}$  is dimensionless.

$\frac{\hat{h}}{\tilde{u}} \gg 1$  suggests: Fluctuations which are large  
w.r.t. the nominal function  
do not cause failure.

suggests: plausibly reliable system.

while

$\frac{\hat{h}}{\tilde{u}} \ll 1$  suggests: Fluctuations which are small  
w.r.t. the nominal function  
entail the possibility of failure.

suggests: plausibly risky system.

¶ For instance, consider the energy-bound info-gap model in eq.(2) on p. 5.

The robustness is dimensional:  $[h] = [u]\sqrt{\text{sec}} = \frac{\text{N}\sqrt{\text{s}}}{\text{m}^2}$ .

Suppose  $T$  is a typical time constant of the system. E.g.:

- Decay time.
- Transient duration.
- Lifetime.
- Etc.

$\tilde{u}(t)$  is a typical  $u$ -function.

So  $\tilde{u}(t)\sqrt{T}$  is a typical-valued function

whose dimensions are the same as  $\hat{h}$ :

$$[\hat{h}] = [\tilde{u}(t)\sqrt{T}].$$

Thus  $\frac{\hat{h}}{\tilde{u}(t)\sqrt{T}}$  is dimensionless.

If  $\frac{\hat{h}}{\tilde{u}(t)\sqrt{T}} \gg 1$  then all functions “much larger” than typical  
do not cause failure.

$\hat{h}$  is plausibly large enough;

The system is plausibly reliable.

If  $\frac{\hat{h}}{\tilde{u}(t)\sqrt{T}} \ll 1$  then there are functions “close” to typical  
which cause failure.

$\hat{h}$  is plausibly too small;

The system is plausibly risky.

¶ Calibration by **dimensional normalization**

is a special case of **reasoning by analogy**.

Recall that an analogical inference is:

Things which are similar in some respects  
will tend to be similar in other respects.

¶ For example, when we concluded that:

$$\frac{\hat{h}}{\tilde{u}(t)\sqrt{T}} \gg 1 \text{ implies 'high robustness'}$$

what analogy are we making?

1. Since  $\frac{\hat{h}}{\tilde{u}(t)\sqrt{T}} \gg 1$

Only loads which are:

very different from  $\tilde{u}(t)$

cause failure.

2. Such loads seem rare, extraordinary, unusual.

3. Thus, by analogy, failure seems rare, extraordinary, unusual  
and the system seems reliable.

## 4 Analogical Reasoning

(Section 4.2)

¶ Now we consider more rigorously the problem of **calibrating**  $\hat{h}(q, r_c)$ .

¶ Stating the problem more quantitatively:

$\hat{h}(q, r_c)$  depends on the decision vector  $q$ .

Let  $q_1$  and  $q_2$  be two different decisions, for which:

$$\hat{h}(q_1, r_c) > \hat{h}(q_2, r_c).$$

This means that, in terms of robustness:

$q_1$  is **better than**  $q_2$ .

But by how much?

Is  $q_1$  substantially, **qualitatively better** than  $q_2$ ?

We seek an evaluation in **linguistic terms**

of **quantitative** increments in  $\hat{h}(q, r_c)$ .

¶ **Analogy, metaphor and translation:**

- We want to translate  $\hat{h}$  or  $\Delta\hat{h}$  from math to natural human language.
- We want to extend the meaning of  $\hat{h}$  or  $\Delta\hat{h}$ .
- Metaphor extends meaning. E.g. “river of time”.
- Analogical reasoning is metaphor between two languages.

¶ For any two entities,  $f_1$  and  $f_2$ ,

If  $f_1$  is “**significantly greater**” than  $f_2$

we will write:  $f_1 \succ f_2$ .

These entities may be **numbers**

or **words**: linguistic values.

Linguistic example:  $f$  = reliability of a system.

where  $f$  takes the values “low”, “moderate”, “high”, etc.

Then “high”  $\succ$  “low”.

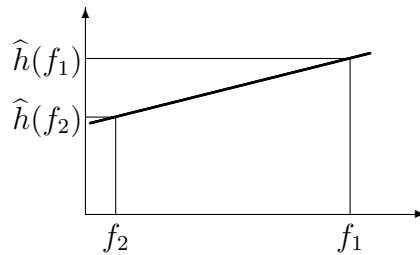


Figure 1: Illustration of calibration by analogy.

¶ Now suppose that the robustness function  $\hat{h}(f)$  increases with  $f$  (see fig. 1):

$$f_1 \succ f_2 \tag{4}$$

and:

$$\hat{h}(f_1) > \hat{h}(f_2) \tag{5}$$

These relations **analogically imply** that:

$$\hat{h}(f_1) \succ \hat{h}(f_2) \tag{6}$$

¶ Eqs.(4)–(6) are an **inference by analogy**:

1. The  $f$  values are **ranked** by significant increments.
2. The  $f$  values and the  $\hat{h}$  values agree in **ordinal ranking**.
3. Hence the  $\hat{h}$  values **tend to be ranked** by significant increments.

¶ Recall that an **analogical inference** is:

Things which **are** similar in **some** respects,  
will **tend** to be similar in **other** respects.

Analogy is **not proof**.

Analogy is **plausible explanation**.

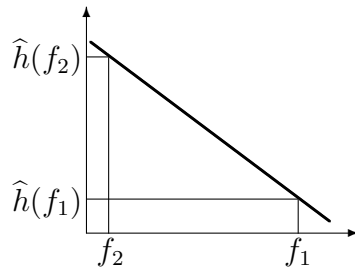


Figure 2: Illustration of calibration by reverse analogy.

¶ There is also reasoning by **reverse analogy**. (see fig. 2)

Instead of: (4) and (5) analogically imply (6).

We have:

$$f_1 \succ f_2 \tag{7}$$

and

$$\hat{h}(f_2) > \hat{h}(f_1) \tag{8}$$

analogically imply that:

$$\hat{h}(f_2) \succ \hat{h}(f_1) \tag{9}$$

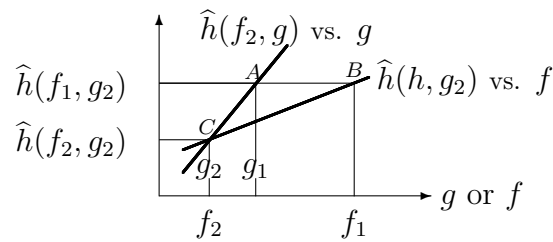


Figure 3: Illustration of calibration by analogy with two variables.

¶ We can also **concatenate** analogical inferences (fig. 3).

Suppose that  $\hat{h}$  depends on two variables:  $\hat{h}(f, g)$ .

1. First analogical inference:

$$f_1 \succ f_2 \text{ and } \hat{h}(f_1, g_2) > \hat{h}(f_2, g_2) \tag{10}$$

analogically imply that:

$$\underbrace{\hat{h}(f_1, g_2)}_B \succ \underbrace{\hat{h}(f_2, g_2)}_C \tag{11}$$

2. Suppose  $\hat{h}(f_2, g)$  increases with  $g$ .

Suppose that  $g_1$  is a value such that:

$$\underbrace{\hat{h}(f_2, g_1)}_A = \underbrace{\hat{h}(f_1, g_2)}_B \tag{12}$$

3. Second analogical inference: Inference (11) ( $B \succ C$ ) with:

$$\underbrace{\hat{h}(f_2, g_1)}_A = \underbrace{\hat{h}(f_1, g_2)}_B \text{ and } g_1 > g_2 \tag{13}$$

analogically imply that:

$$g_1 \succ g_2 \tag{14}$$

¶ We now consider an example of analogical inference for calibrating  $\hat{h}$  by **severity of consequences**.

## 5 Calibration by Consequence Severity

¶ Surface treatment problem:

- Determine program of treatment for a region.  
E.g. apply protective material, apply fertilizer, allow grazing, etc.
- Use model of influence of treatment on the region.
- Model is uncertain.
- Determine the treatment to satisfy the level of influence.
- Evaluate robustness function.
- Numerical values of influence are understood linguistically.
- Calibrate robustness function to find significant increments of  $\hat{h}$ .

¶ First we evaluate  $\hat{h}(q, r_c)$ .

Then we use analogical reasoning by consequence severity  
to calibrate  $\hat{h}(q, r_c)$ .

### 5.1 Robustness function

¶ Basic terms:

$x$  = position in region.  $0 \leq x \leq 1$ .

$q(x)$  = density of treatment at  $x$ .

$b(x)$  = uncertain influence function.

$R(q, b)$  = benefit to region due to treatment. Numerical values understood qualitatively.

¶ Model of benefit function:

$$R(q, b) = \int_0^1 b(x)q(x) dx \quad (15)$$

$b(x)$  = uncertain influence function.

= benefit at  $x$  from unit treatment at  $x$ .

Note: the **model**  $R(q, b)$  for how treatment  $q$  influences the region is **uncertain**.

¶ What we know about the uncertain part of the model,  $b(x)$ :

- The nominal function,  $\tilde{b}(x)$ .
- $b(x)$  deviates from  $\tilde{b}(x)$ :
  - The deviation is not erratic.
  - Sudden shifts in  $u(x)$  are not plausible.
- Variation of  $b(x)$  is greatest in the mid-region.
- Variation of  $b(x)$  vanishes at the boundaries.

¶ Use an envelope-slope-bound info-gap model:

$$\mathcal{U}(h, \tilde{b}) = \left\{ b(x) = \tilde{b}(x) + u(x) : u(0) = u(1) = 0, \left| \frac{du(x)}{dx} \right| \leq h\psi(x) \right\}, \quad h \geq 0 \quad (16)$$

where  $\psi(x)$  is the known envelope function.

¶ Critical performance requirement:

$$R(q, b) \geq r_c \quad (17)$$

¶ Robustness function:

$$\hat{h}(q, r_c) = \max \left\{ h : \left( \min_{b \in \mathcal{U}(h, \tilde{b})} R(q, b) \right) \geq r_c \right\} \quad (18)$$

¶ The benefit function, eq.(15) on p.14, can be written:

$$R(q, b) = \underbrace{\int_0^1 \tilde{b}(x)q(x) dx}_{\tilde{R}(q)} + \int_0^1 u(x)q(x) dx \quad (19)$$

$\tilde{R}(q)$  is the nominal, error-free benefit from treatment  $q(x)$ .

Define the following known function:

$$\eta(x) = \int_0^x q(x') dx' \quad \implies \quad d\eta(x) = q(x)dx \quad (20)$$

The right-most integral in eq.(19) is found by integration by parts ( $\int_a^b gdf = gf|_a^b - \int_a^b f dg$ ):

$$\int_0^1 u(x)q(x) dx = \int_0^1 u(x) d\eta(x) \quad (21)$$

$$= \underbrace{u(x)\eta(x)|_0^1}_0 - \int_0^1 \eta(x) \frac{du(x)}{dx} dx \quad (22)$$

Hence the minimum benefit up to info-gap  $h$  is:

$$\min_{b \in \mathcal{U}(h, \tilde{b})} R(q, b) = \tilde{R}(q) - h \int_0^1 |\eta(x)| \psi(x) dx \quad (23)$$

Equating this to  $r_c$  and solving for  $h$  yields  $\hat{h}$ :

$$\hat{h}(q, r_c) = \frac{\tilde{R}(q) - r_c}{\int_0^1 |\eta(x)| \psi(x) dx} \quad (24)$$

unless this is negative, in which case  $\hat{h}(q, r_c) = 0$ .

## 5.2 Special Case

¶ Assumptions:

$q$  = density of treatment = constant over  $x \in [0, 1]$ .

$\tilde{b}(x) = b_0 + b_1 \sin \pi x$  = known nominal influence function.

$\psi(x) =$  known envelope function =  $\sin \pi x$ .

¶ With these assumptions the robustness function becomes:

$$\hat{h}(q, r_c) = \pi b_0 + 2b_1 - \frac{\pi r_c}{q} \tag{25}$$

Eq.(25) shows that, at fixed  $r_c$ ,  $\hat{h}(q, r_c)$  **increases** with increasing  $q$ .

Eq.(25) shows that, at fixed  $q$ ,  $\hat{h}(q, r_c)$  **decreases** with increasing  $r_c$ .

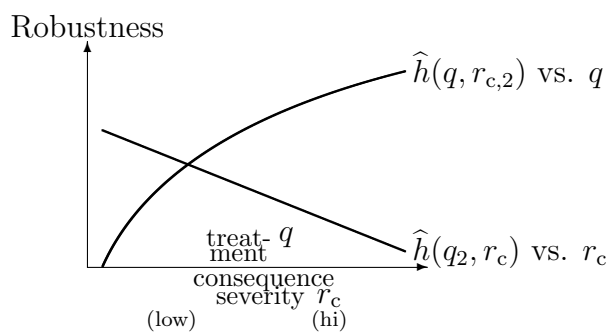


Figure 4: Robustness  $\hat{h}(q, r_c)$  vs.  $r_c$  and vs.  $q$ , eq.(25).

¶ Questions:

- What is a significant increment in  $\hat{h}$ ?
- What is a significant increment in  $q$ ?

### 5.3 Calibration by Consequence Severity

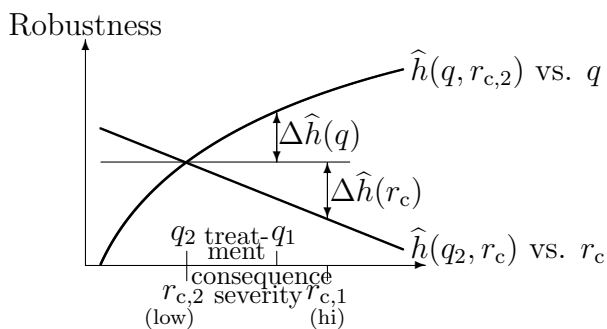


Figure 5: Robustness  $\hat{h}(q, r_c)$  vs.  $r_c$  and vs.  $q$ , eq.(25).

¶ We answer these questions, for the special case, by consequence-severity calibration.

¶ Fig. 5 shows two robustness curves.

- The decreasing straight line shows trade-off of  $\hat{h}$  vs.  $r_c$ .
- The increasing curve shows augmented  $\hat{h}$  resulting from increased treatment  $q$ .
- Two values of reward (benefit) are shown:  $r_{c,1} > r_{c,2}$ .
- These levels of reward are understood intuitively:

$r_{c,1}$  is substantially greater benefit than  $r_{c,2}$ :

$$r_{c,1} \succ r_{c,2}.$$

$r_{c,1}$  is “high” benefit.

$r_{c,2}$  is “low” benefit.

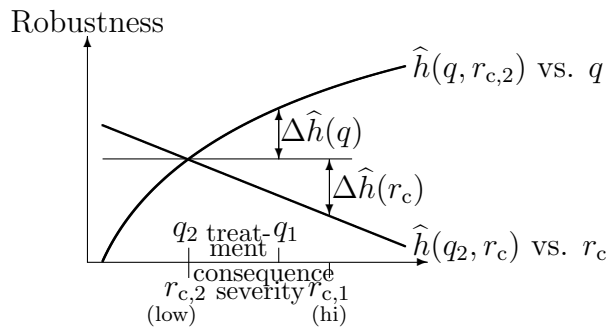


Figure 6: Robustness  $\hat{h}(q, r_c)$  vs.  $r_c$  and vs.  $q$ , eq.(25).

¶ Now concatenate 2 analogical inferences in fig. 6.

- First inference: fixed treatment  $q_2$ :

$\Delta\hat{h}(r_c)$  = increment in robustness from  $r_{c,1}$  to  $r_{c,2}$ .

By reverse analogical inference:

$\Delta\hat{h}(r_c)$  = significant increment in robustness.

That is:

$r_{c,1} \succ r_{c,2}$  and  $\hat{h}(q_2, r_{c,2}) > \hat{h}(q_2, r_{c,1})$  implies

$\hat{h}(q_2, r_{c,2}) \succ \hat{h}(q_2, r_{c,1})$ .

That is:  $\Delta\hat{h}(r_c)$  is “large”.

- 2nd inference.

$\Delta\hat{h}(q)$  = increment in robustness from  $q_2$  to  $q_1$  at fixed benefit  $r_{c,2}$ .

Choose  $q_1$  so that  $\Delta\hat{h}(q) = \Delta\hat{h}(r_c)$

Now, with 1st inference ( $\Delta\hat{h}(r_c)$  is “large”): we conclude:

$\hat{h}(q_1, r_{c,2}) \succ \hat{h}(q_2, r_{c,2})$ .

Thus the increment in treatment from  $q_2$  to  $q_1$

is significant in analogy to the corresponding increment in  $\hat{h}$ .

¶ Summary of this analogical argument:

1. The increment  $\Delta h(q)$  from  $q_2$  to  $q_1$  (at fixed  $r_c$ ) agrees in robustness to the increment  $\Delta h(r_c)$  from  $r_{c,1}$  to  $r_{c,2}$ .
2. The increment  $\Delta h(r_c)$  from  $r_{c,1}$  to  $r_{c,2}$  is a qualitatively significant change in benefit, by analogy to  $r_{c,2} \succ r_{c,1}$ .
3. Thus the increment in treatment from  $q_2$  to  $q_1$  is expected, by analogy, to be significant.