2. **Non-linear spring.** (p.13) An asymmetric strain-hardening non-linear spring has the following relation between displacement x and force f:

$$f = k_1 x + k_2 x^2 \tag{2}$$

Both coefficients, k_1 and k_2, are positive.

The spring is permanently damaged if the magnitude of the displacement exceeds x_c, and you require the probability of failure not to exceed P_c.

The magnitude of the force is uncertain and described by an incompletely known probability density. Our best probabilistic model is that the probability density of the force f is uniform on the interval $[0, F]$, where F is a known value. However, there is evidence that rare events of greater force can occur. That is, there is a tail of the probability distribution for $f > F$, but the shape and magnitude of this tail is entirely unknown. In other words, the probability density is:

$$p(f) = \begin{cases} \text{constant}, & 0 \le f \le F \\ \text{variable}, & F \le f \end{cases} \tag{3}$$

The value of F is known, but the behavior of $p(f)$ for $F \le f$ is entirely unknown. Derive an expression for the robustness of predicting the probability of failure of the spring to uncertainty in the loading force.

Solution to Problem 2.

The first question we must consider is: how to model the uncertainty in the pdf of the force f. What we **do know**:

- ∘ f is non-negative.
- ∘ $p(f)$ is constant for $0 \le f \le F$.
- ∘ The value of F.

What we **do not know**:

- ∘ The actual constant value of $p(f)$ for $0 \le f \le F$.
- ∘ The behavior of $p(f)$ for $f > F$.

This very severe info-gap could be modelled as:

$$\mathcal{U}(h) = \left\{ p(f): \; p(f) \ge 0; \; p(f) = \frac{1}{F}\left(1 - \int_F^\infty p(f)\,\mathrm{d}f\right), \; 0 \le f \le F; \; \int_F^\infty p(f)\,\mathrm{d}f \le h \right\}, \quad h \ge 0 \tag{66}$$

The system fails if:

$$x \ge x_\mathrm{c} \tag{67}$$

Given a pdf, the probability of failure of the system is:

$$P_\mathrm{f}(p) = \mathrm{Prob}(x \ge x_\mathrm{c} | p) \tag{68}$$

Of course, we cannot evaluate this probability because we do not know $p(f)$.

f, k_1 and k_2 are positive. Using eq.(2), the condition for failure, eq.(67), can be expressed as:

$$f \ge {x_\mathrm{c}}^2 k_2 + x_\mathrm{c} k_1 = \phi \tag{69}$$

which defines ϕ. Thus the probability of failure is:

$$P_\mathrm{f}(p) = \mathrm{Prob}(f \ge \phi | p) \tag{70}$$

We require:

$$P_\mathrm{f}(p) \le P_\mathrm{c} \tag{71}$$

The robustness with respect to uncertainty in $p(f)$ is:

$$\widehat{h}(P_\mathrm{c}) = \max\left\{ h: \; \max_{p\in\mathcal{U}(h)} P_\mathrm{f}(p) \le P_\mathrm{c} \right\} \tag{72}$$

If $\phi \le F$, then:

$$P_\mathrm{f}(p) = \underbrace{(F-\phi)\frac{1}{F}\left(1 - \int_F^\infty p(f)\,\mathrm{d}f\right)}_{\text{Failure in known part}} + \underbrace{\int_F^\infty p(f)\,\mathrm{d}f}_{\text{Failure on unknown tail}} \tag{73}$$

$$= 1 - \frac{\phi}{F} + \frac{\phi}{F}\int_F^\infty p(f)\,\mathrm{d}f \tag{74}$$

Thus:

$$\max_{p\in\mathcal{U}(h)} P_\mathrm{f}(p) = 1 - \frac{\phi}{F} + \frac{\phi}{F}h \tag{75}$$

$\widehat{h}(P_\mathrm{c}) = 0$ if $1 - \frac{\phi}{F} > P_\mathrm{c}$ or, equivalently, if $\frac{\phi}{F} < 1 - P_\mathrm{c}$.

Equating eq.(75) to P_c and solving for h yields the robustness, (still assuming $\phi \le F$):

$$\widehat{h}(P_\mathrm{c}) = \begin{cases} 1 - \dfrac{1-P_\mathrm{c}}{\phi/F} & \text{if } \dfrac{\phi}{F} \ge 1 - P_\mathrm{c} \\ 0 & \text{else} \end{cases} \tag{76}$$

If $\phi > F$ then:

$$P_{\rm f}(p) = \int_{\phi}^{\infty} p(f)\,{\rm d}f \tag{77}$$

So:

$$\max_{p \in \mathcal{U}(h)} P_{\rm f}(p) = h \tag{78}$$

Hence:

$$\widehat{h} = P_{\rm c} \tag{79}$$

Fig. 1 shows that $\widehat{h}(P_{\rm c})$ increases as $P_{\rm c}$ increases. That is, we are more robust (to uncertainty in the pdf of the load) in predicting large probabilities of failure than in predicting low probabilities of failure.

Fig. 2 shows that $\widehat{h}(P_{\rm c})$ increases as the failure threshold ϕ increases, at fixed probability of failure $P_{\rm c}$.

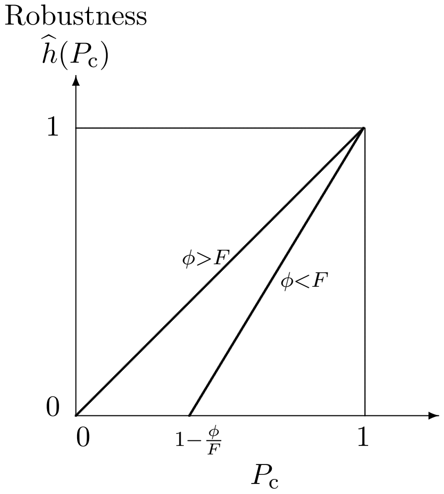

Figure 1: Robustness vs. failure probability.

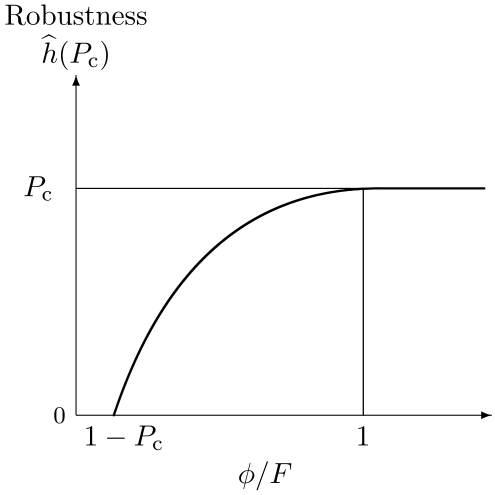

Figure 2: Robustness vs. failure threshold.

Combining eqs.(76) and (79) we see that we **improve $\widehat{h}$ by increasing ϕ.** Let us then consider the choice of the two stiffness coefficients, k_1 and k_2.

$$\frac{\partial \phi}{\partial k_1} = x_{\rm c} > 0 \tag{80}$$

$$\frac{\partial \phi}{\partial k_2} = x_{\rm c}^2 > 0 \tag{81}$$

Thus ϕ is increased and $\widehat{h}$ is improved by increasing either or both k_1 and k_2. Making the system stiffer decreases the probability of failure, and increases the reliability with which we predict the probability of failure.