Problem Set on Hybrid Uncertainties

List of problems:

1. **Surface treatment.** Many design and decision problems involve surface treatments of various sorts: metallic coatings to prevent corrosion, submarine coatings to reduce sonar reflection, tribological treatments to reduce or to enhance friction, horticultural treatment to manage parasites. In this problem we examine a generic surface treatment problem.

   You are the responsible for maintaining the functionality of a surface. The surface is subject to occasional attacks, e.g. by corrosive materials, and it is your responsibility to establish a maintenance strategy of spraying and other treatments. These treatments are needed both to maintain functionality and to minimize the cumulative damage of the attacks over a unit of time (e.g., a year). The treatment policy is expressed as a function $q(x)$ over the surface. $q(x)$ is the intensity of treatment at point x. A specified total quantity of material Q must be applied to the surface each year; the question is how to distribute this material.

   Attacks recur randomly and independently in time, with an average rate of λ per year. However, the infestation density, $u(x)$ per square meter, is never uniform over the surface. Some areas are known to suffer more, and other areas are known to display wide fluctuations in the density of attack. The damage to the surface from a single attack is:

   $$\delta(u) = \int q(x)u(x)\,\mathrm{d}x \qquad (1)$$

   The surface is in danger of not recovering if the cumulative damage from all the attacks of the year exceeds the critical value of Δ_{c}. Develop an expression for the robustness to the uncertainty in spatial distribution and temporal recurrence of attacks.

2. **Non-linear spring.** An asymmetric strain-hardening non-linear spring has the following relation between displacement x and force f:

   $$f = k_1 x + k_2 x^2 \qquad (2)$$

   Both coefficients, k_1 and k_2, are positive.

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The spring is permanently damaged if the magnitude of the displacement exceeds $x_{\rm c}$, and you require the probability of failure not to exceed $P_{\rm c}$.

The magnitude of the force is uncertain and described by an incompletely known probability density. Our best probabilistic model is that the probability density of the force f is uniform on the interval $[0,\, F]$, where F is a known value. However, there is evidence that rare events of greater force can occur. That is, there is a tail of the probability distribution for $f > F$, but the shape and magnitude of this tail is entirely unknown. In other words, the probability density is:

$$p(f) = \begin{cases} \text{constant}, & 0 \le f \le F \\ \text{variable}, & F \le f \end{cases} \tag{3}$$

The value of F is known, but the behavior of $p(f)$ for $F \le f$ is entirely unknown. Derive an expression for the robustness of predicting the probability of failure of the spring to uncertainty in the loading force.

3. **Static deflection of a cantilever.** Consider a uniform cantilever beam subject to N static point loads applied perpendicular to the beam axis. The loads all lie in a single plane. The vector of loads is denoted:

$$f = (f_1,\, \ldots,\, f_N)^T \tag{4}$$

The deflection of the free end of the cantilever, y, is linearly related to the loads by:

$$y = k^T f \tag{5}$$

where k is a column vector of known stiffness coefficients.

The load vector is uncertain and belongs to the following ellipsoid bound info-gap model:

$$F(h, \widetilde{f}) = \left\{ f : \ \left(f - \widetilde{f}\right)^T W \left(f - \widetilde{f}\right) \le h^2 \right\}, \quad h \ge 0 \tag{6}$$

where W is a real, symmetric, positive definite matrix.

The beam fails if the end deflection exceeds the critical value $y_{\rm c}$. Let the uncertainty parameter h in the info-gap model be a random variable with a known exponential distribution. Combine this information with the info-gap model of uncertainty about the load vector f to evaluate an upper bound on the probability of failure.

4. **Simple harmonic oscillator.** (p.13) The displacement $x(t)$ of an oscillator in simple harmonic motion is:

$$m\ddot{x}(t) + kx(t) = u(t) \tag{7}$$

where the mass m and stiffness k are known but the driving force $u(t)$ is uncertain. Assume that the initial displacement and velocity of the oscillator are both zero.

The energy of the oscillator is proportional to the square of the displacement. Failure occurs if the energy at a specified time T exceeds a critical value:

$$x^2(T) \ge E_{\rm c} \tag{8}$$

The uncertainty in $u(t)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$u(t) = \widetilde{u}(t) + \sum_{n=1}^{N} \phi_n \sin \frac{n\pi t}{T} \tag{9}$$

$$= \widetilde{u}(t) + \phi^T \sigma(t) \tag{10}$$

where ϕ is the vector of uncertain Fourier coefficients and $\sigma(t)$ is the vector of corresponding sine functions. The info-gap model is:

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u(t) = \widetilde{u}(t) + \phi^T\sigma(t) : \ \phi^T W \phi \le h^2 \right\}, \quad h \ge 0 \tag{11}$$

where W is a known, real, symmetric positive definite matrix.

Let the uncertainty parameter h have a known normal distribution. Evaluate an upper bound on the probability of failure.

5. **A suspicious gamble.** (p.15) A suspicious character offers you to play a game of chance. On each throw of a coin which he provides you will receive a positive reward v_1 if the outcome is "heads" and no reward otherwise. The character claims that the probability of "heads" is 0.5, but the fractional error of this claim is unknown, though it is certainly not more than a few tens of percent of the nominal value. You have been offered to participate in a long series of throws of the coin, where each throw costs v_c dollars. Thus it is important that the average return on a throw equal no less than v_c.

   (a) Choose an info-gap model for uncertainty in the probability of "heads".

   (b) Develop an explicit analytical expression for the robustness of a single throw.

   (c) Based on your experience with suspicious characters, you judge that the fractional error of the claimed 0.5 probability of "heads" is not more than a few tens of percent of the nominal value. If you require robustness of 0.5, what is the greatest price of a single throw that you are willing to pay, expressed as a fraction of v_1?

6. **Braking system.** A mechanical braking mechanism has a internal force f, which is uncertain. The braking force is given by:

$$g = \mu L f \tag{12}$$

   We require that the braking force exceed the critical value g_c:

$$g \ge g_c \tag{13}$$

   The internal force is given by:

$$f = f_0 - u \tag{14}$$

   where f_0 is constant, positive and known, and u is exponentially distributed:

$$p(u) = \lambda e^{-\lambda u}, \quad u \ge 0 \tag{15}$$

   (a) Calculate the probability that the system will satisfy the requirement in eq.(13).

   (b) Now suppose that the coefficient λ of the exponential distribution is uncertain:

$$\mathcal{U}(h,\widetilde{\lambda}) = \left\{ \lambda : \ \lambda > 0, \ |\lambda - \widetilde{\lambda}| \le h\widetilde{\lambda} \right\}, \quad h \ge 0 \tag{16}$$

   It is required that the probability of satisfying eq.(13) be no less than P_c. Calculate the robustness function.

7. **First-failure testing with uncertain sample.** (p.16) We will sample a population of items and test each item for integrity. Each item will either pass (P) or fail (F). We will stop sampling when the first F occurs.

(a) What is the probability that the sample will stop on the xth test, if the fraction of F's in the population is p? What is the average and standard deviation of the sample size?

(b) We have sampled N items without finding an F. Use the sampling distribution derived in the previous step to test the null hypothesis:

$$H_0: \quad p = 0.01 \tag{17}$$

against the alternative hypothesis:

$$H_1: \quad p < 0.01 \tag{18}$$

(c) Now suppose that we are unsure the sample is statistically random. Alternatively, we are unsure if the population which we sampled is the population we actually want to test. Thus we are not sure that the sample distribution is in fact the distribution derived in the first step. Call the distribution in part (a) $\widetilde{p}(x)$. An info-gap model for uncertainty in the distribution is:

$$\mathcal{U}(h, \widetilde{p}) = \left\{ p(x): \ p(x) \geq 0, \ \sum_{x=1}^{\infty} p(x) = 1, \ |p(x) - \widetilde{p}(x)| \leq h\widetilde{p}(x), \ \forall x \right\}, \quad h \geq 0 \tag{19}$$

The robustness for rejecting H_0, at significance level h, is the greatest horizon of uncertainty up to which H_0 is rejected at this significance. Formulate and derive the robustness function, assuming that N is much larger than the mean.

8. **Two lotteries.** (p.17) You must choose between two options, where each option has a random cost, c, between 0 and 1. The estimated pdf of the cost for the first option is constant, $\widetilde{p}(c) = 1$, but this pdf is uncertain, and the info-gap model is:

$$\mathcal{U}(h) = \left\{ p(c): \ p(c) \geq 0, \ \int_0^1 p(c)\, \mathrm{d}c = 1, \ |p(c) - 1| \leq h \right\}, \quad h \geq 0 \tag{20}$$

We don't know the size of the error of $\widetilde{p}_1(c)$, but it might well be on the order of ten or twenty percent, or perhaps more.

The pdf of the second option is known precisely and is $\widetilde{p}_2(c) = 2c$.

The expected loss with the second option (based on $\widetilde{p}_2(c)$) exceeds the expected loss with the first option (based on $\widetilde{p}_1(c)$), but $\widetilde{p}_1(c)$ is uncertain. Use an info-gap robust-satisficing analysis to select between these two options.

9. **Uncertain pdf.** (p.18) The response of a system, x, is a random variable in the interval $[1, \ \infty)$. The estimated pdf is:

$$\widetilde{p}(x) = \frac{1}{x^2}, \quad x \geq 1 \tag{21}$$

The uncertainty in the pdf is represented with an info-gap model:

$$\mathcal{U}(h) = \left\{ p(x): \ p(x) \geq 0, \ \int_1^{\infty} p(x)\, \mathrm{d}x = 1, \ |p(x) - \widetilde{p}(x)| \leq h \right\}, \quad h \geq 0 \tag{22}$$

It is required that the response of the system be no greater than x_{c} with probability no less than P_{c}. Derive an explicit algebraic expression for the robustness.

10. **Two discrete lotteries.** (p.18) You must choose between two options, where each option has a random reward, r, which equals either -1 or $+1$. The estimated probability distribution of the reward for the first option is:

$$\widetilde{p}_1(r) = \begin{cases} 0.2 & \text{if } r = -1 \\ 0.8 & \text{if } r = +1 \end{cases} \quad (23)$$

However, this distribution is uncertain as represented by the following info-gap model:

$$\mathcal{U}(h) = \{p_1(r) : \ 0 \le p_1(1) \le 1, \ p_1(-1) + p_1(1) = 1, \ |p_1(r) - \widetilde{p}_1(r)| \le h\}, \ h \ge 0 \quad (24)$$

The probability distribution of reward from the second option is known precisely and is:

$$p_2(r) = \begin{cases} 0.25 & \text{if } r = -1 \\ 0.75 & \text{if } r = +1 \end{cases} \quad (25)$$

The expected rewards from the two options are:

$$\mathrm{E}(r|\widetilde{p}_1) = -1 \times 0.2 + 1 \times 0.8 = 0.6 \quad (26)$$

$$\mathrm{E}(r|p_2) = -1 \times 0.25 + 1 \times 0.75 = 0.5 \quad (27)$$

The first option looks better (higher expected return) but is more uncertain (its probability distribution is imprecise). Use an info-gap robust-satisficing analysis to select between these two options.

11. **Serial network.** (p.19) Consider a serial network with n subunits, each of which is essential for the operation of the system. The subunits are assumed to fail independently. The estimated probability of failure of the ith subunit is $\widetilde{p}_i$ with error estimate s_i. An info-gap model for uncertainty in these probabilities is:

$$\mathcal{U}(h) = \left\{p : \ p_i \in [0, \ 1], \ \left|\frac{p_i - \widetilde{p}_i}{s_i}\right| \le h, \ \forall i\right\}, \quad h \ge 0 \quad (28)$$

The reliability of the network—probability of no failure—is:

$$R(p) = \prod_{i=1}^{n} (1 - p_i) \quad (29)$$

(a) *n serial subunits.* We require reliability no less than R_c. Derive the robustness function. Consider a serial network with 50 identical independent subunits, each with estimated probability of failure of $\widetilde{p}_i = 0.001$ and estimated error of $s_i = 0.0001$. What probabilities of failure have robustnesses of 0, 1, 2 and 3? Explain the meaning of these results.

(b) *One subunit, $n = 1$, two designs.* Compare two different designs whose properties are:

$$\frac{s}{s'} < \frac{1 - \widetilde{p}}{1 - \widetilde{p}'} < 1 \quad (30)$$

(c) *Two serial subunits, uncertain correlated failures.*[1] Let p_{f1} denote the probability of failure of the first unit, regardless of what happens to the second unit. Likewise, let p_{f2}

[1] This example is a simple version of the analysis of the junior tranche in section 4.1 of Ben-Haim, *Info-Gap Economics: An Operational Introduction.*

denote the probability of failure of the second unit, regardless of what happens to the first unit. We assume that $p_{\rm f1}$ and $p_{\rm f2}$ are known precisely. Finally, let $p_{\rm f12}$ denote the probability of failure of both units, whose estimated value is $\widetilde{p}_{\rm f12}$. We do not know how accurate this estimate is.

The probability of failure of the serial system is:

$$P_{\rm f} = p_{\rm f1} + p_{\rm f2} - p_{\rm f12} \tag{31}$$

Develop an info-gap model for this uncertainty, and derive the robustness function for reliability of the system. Then consider the special case that $p_{\rm f1} = p_{\rm f2} = 0.001$ and $\widetilde{p}_{\rm f12} = 0.0005$.

12. **Parallel network.** (p.21) Consider a parallel network with n subunits, such that the system operates provided that at least one subunit is functional. That is, the subunits are all redundant. The subunits are assumed to fail independently. The estimated probability of failure of the ith subunit is $\widetilde{p}_i$ with error estimate s_i. An info-gap model for uncertainty in these probabilities is:

$$\mathcal{U}(h) = \left\{ p : \ p_i \in [0,\ 1], \ \left| \frac{p_i - \widetilde{p}_i}{s_i} \right| \le h, \ \forall i \right\}, \quad h \ge 0 \tag{32}$$

The reliability of the network—probability of no failure—is:

$$R(p) = 1 - \prod_{i=1}^{n} p_i \tag{33}$$

(a) *n parallel subunits.* We require reliability no less than $R_{\rm c}$. Derive the robustness function. Consider a parallel network with 3 identical independent subunits, each with estimated probability of failure of $\widetilde{p}_i = 0.1$ and estimated error of $s_i = 0.03$. What probabilities of failure have robustnesses of 0, 1, 3, 5 and 10? Explain the meaning of these results.

(b) *One subunit, $n = 1$, two designs.* Compare two different designs whose properties are:

$$\frac{s}{s'} < \frac{1 - \widetilde{p}}{1 - \widetilde{p}'} < 1 \tag{34}$$

Compare the results to the solution of part (b) of problem 11.

13. **Time to Failure.** (p.21) Consider a system whose time to failure is a random variable, t. The estimated probability density function (pdf) is exponential:

$$\widetilde{p}(t) = \lambda {\rm e}^{-\lambda t} \tag{35}$$

The reliability of the system for the duration from initiation (at time 0) to time t is the probability that failure will occur after t.

(a) The pdf is known to be exponential but its estimated coefficient, $\widetilde{\lambda}$, is uncertain, as represented by an info-gap model:

$$\mathcal{U}(h) = \left\{ p(t) = \lambda {\rm e}^{-\lambda t} : \ \lambda \ge 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \le h \right\}, \quad h \ge 0 \tag{36}$$

Derive an expression for the robustness to uncertainty in λ given the requirement that the reliability exceed $R_{\rm c}$.

Evaluate and discuss the meaning of several robustness values for $\widetilde{\lambda} = 0.001/\text{hr}$ and $t = 100\text{hr}$.

(b) The pdf is thought to be exponential but its actual shape is highly uncertain. More fundamentally, the exponential distribution derives from assumptions about the failure mechanisms—e.g. independence of events—which may not hold. The uncertainty in the pdf is represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ p(t) : \ p(t) \geq 0, \ \int_0^\infty p(t) \, \mathrm{d}t = 1, \ |p(t) - \widetilde{p}(t)| \leq \widetilde{p}(t) h \right\}, \quad h \geq 0 \tag{37}$$

Derive the robustness function and discuss its meaning.

Supplementary and Background Problems

1. Consider the Poisson distribution:

$$P_n = \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!}, \quad n = 0,\, 1,\, 2,\, \ldots \tag{38}$$

(a) Show that P_n is normalized.

(b) Find the first moment of P_n: the average value of n

(c) Find the first moment of P_n, conditioned on the fact that $n \ge r$.

2. Consider the following info-gap model:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(x) : \ |u(x)| \le h\psi(x) \right\}, \quad h \ge 0 \tag{39}$$

(a) What $u \in \mathcal{U}(h, \widetilde{u})$ maximizes $\int_0^{2\pi} u(x) \sin x \, \mathrm{d}x$? What is the resulting maximum?

(b) What $u \in \mathcal{U}(h, \widetilde{u})$ maximizes $\int_0^{2\pi} u(x) |\sin x| \, \mathrm{d}x$? What is the resulting maximum?