6. **Static deflection of a cantilever: continued.** (p.45) In some problems the reward function can be expressed as a quadratic function of an info-gap uncertain vector. We modify problem 4 as an example. Instead of the deflection y in eq.(14), which is a linear performance function, let us consider:

$$r = f^T V f \tag{21}$$

r is proportional to an elastic energy of deformation. V is a known, real, symmetric, positive definite matrix.

Re-examine questions 4a–4c for the following two cases: **(a)** $\widetilde{f} = 0$. **(b)** $\widetilde{f} \neq 0$.

Solution for problem 6. (p.6)

(a) To optimize $f^T V f$ subject to $f^T W f \leq h^2$ we use Lagrange optimization. Define:

$$H = f^T V f + \lambda \left(h^2 - f^T W f \right) \tag{280}$$

Extrema occur at:

$$\frac{\partial H}{\partial f} = 2Vf - 2\lambda W f \quad \Longrightarrow \quad (V - \lambda W) f = 0 \tag{281}$$

This can be re-written as:

$$W^{1/2}(W^{-1/2} V W^{-1/2} - \lambda I) W^{1/2} f = 0 \tag{282}$$

which is equivalent to:

$$(W^{-1/2} V W^{-1/2} - \lambda I) W^{1/2} f = 0 \tag{283}$$

Define $\Xi = W^{-1/2} V W^{-1/2}$. We see that f is a solution of eq.(283) if $W^{1/2} f$ is an eigenvector of Ξ. Denote the orthonormal eigenvectors of Ξ, and the corresponding eigenvalues, by $\xi_1, \ldots, \xi_N$ and $0 < \mu_1 \leq \cdots \leq \mu_N$. Thus:

$$\Xi \xi_i = \mu_i \xi_i \quad \text{and} \quad \xi_i^T \xi_j = \delta_{ij} \tag{284}$$

Thus $f = c W^{-1/2} \xi_i$ for some i and c is determined by the constraint:

$$h^2 = f^T W f = c^2 \xi^T W^{-1/2} W W^{-1/2} \xi = c^2 \xi_i^T \xi_i \quad \Longrightarrow \quad c = \pm h \tag{285}$$

Hence:

$$\max f^T V f = \max_i c^2 \xi_i^T W^{-1/2} V W^{-1/2} \xi_i = \max_i c^2 \xi_i^T \Xi \xi_i = \max_i c^2 \mu_i \xi_i^T \xi_i = h^2 \mu_N \tag{286}$$

Equate this to the critical value of r to find the robustness:

$$\widehat{h} = \sqrt{\frac{r_{\rm c}}{\mu_N}} \tag{287}$$