47. **Extrapolating an S–N curve,** (p.97). An S–N curve is the functional relation between the load amplitude S and the number of cycles to fatigue failure, N. A (very simple and inaccurate) model is:

$$S(N) = \frac{a}{b+N} \tag{145}$$

(a) Given observed cycles to failure with corresponding load amplitudes, (N_i, S_i), $i = 1, \ldots, K$, derive a least-squares estimate of the coefficient a, assuming that b is known.

(b) Let $N_{\rm max}$ denote the greatest lifetime which has been observed. We want to predict the load which will yield a lifetime $N_0 > N_{\rm max}$ for some specified value of N_0.

Let us suppose that eq.(145) accurately describes the S–N curve. Let us furthermore suppose that the value of a is known precisely. However, we are not sure that the value of b used for lower lifetimes is still valid when we extrapolate. Let us write eq.(145) as $S(N, b)$.

Suppose that b is estimated at $\widetilde{b}$ with approximate error σ_b, but the true value of b is unknown. An info-gap model for uncertainty in b is:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b - \widetilde{b}}{\sigma_b} \right| \le h \right\}, \quad h \ge 0 \tag{146}$$

The true failure load at lifetime N_0 is $S(N_0, b)$, described by eq.(145) with known a and uncertain b. We will extrapolate (predict) the failure load at lifetime N_0 by using eq.(145) with a value for b of our choice, call it $b_{\rm c}$. We require that the true load at lifetime N_0 not exceed the predicted load by more than ε:

$$S(N_0, b) \le S(N_0, b_{\rm c}) + \varepsilon \tag{147}$$

Derive an expression for the robustness of the choice $b_{\rm c}$.

Solution for problem 47. (p.27)

(a) The mean squared error is:

$$G = \frac{1}{K} \sum_{i=1}^{K} \left(S_i - \frac{a}{b + N_i} \right)^2 \tag{816}$$

Differentiating wrt a, equating to zero, and solving for a yields the least squares estimate:

$$a_{\text{LS}} = \frac{\sum_{i=1}^{K} \frac{S_i}{b+N_i}}{\sum_{i=1}^{K} \frac{1}{(b+N_i)^2}} \tag{817}$$

(b) The robustness is defined as:

$$\widehat{h}(b_{\text{c}}, \varepsilon) = \max \left\{ h : \left(\max_{b \in \mathcal{U}(h)} S(N_0, b) \right) \le S(N, b_{\text{c}}) + \varepsilon \right\} \tag{818}$$

Let $\mu(h)$ denote the inner maximum, which is the inverse of the robustness function. This inner maximum occurs when b is minimal at horizon of uncertainty h:

$$\mu(h) = \frac{a}{\widetilde{b} - \sigma_b h + N_0} \tag{819}$$

$$= \frac{1}{\frac{1}{S(N_0, \widetilde{b})} - \frac{\sigma_b}{a} h} \tag{820}$$

Equating this to $S(N_0, b_{\text{c}}) + \varepsilon$ and solving for h yields the robustness:

$$\widehat{h}(b_{\text{c}}, \varepsilon) = \left(\frac{1}{S(N_0, \widetilde{b})} - \frac{1}{S(N, b_{\text{c}}) + \varepsilon} \right) \frac{a}{\sigma_b} \tag{821}$$

or zero if this is negative. Specifically:

$$\widehat{h}(b_{\text{c}}, \varepsilon) = 0 \quad \text{if} \quad \varepsilon \le S(N_0, \widetilde{b}) - S(N, b_{\text{c}}) \tag{822}$$

The robustness vs. ε has positive slope. Also, the robustness curve shifts to the left (robustness improves) as $S(N, b_{\text{c}})$ increases, approaching $S(N_0, \widetilde{b})$ from below. However, as $S(N, b_{\text{c}})$ increases, the robustness curve becomes flatter—lower slope. Thus robustness curves for different b_{c} values can cross one another.