Problem Set on

Robustness and Opportuneness

List of problems:

1. **Robustness and opportunity.** Consider an uncertain scalar function $u(t)$. Adopt the following "minimal requirement" in the definition of robustness:

$$u(t) \ge r_{\rm c} \quad \text{for} \quad 0 \le t \le T \tag{1}$$

Likewise, the condition for "sweeping success" in the definition of the opportunity is chosen to be:

$$u(t) \ge r_{\rm w} \quad \text{for} \quad 0 \le t \le T \tag{2}$$

where $r_{\rm w}$ is greater, usually much greater, than $r_{\rm c}$.

For each of the info-gap models listed below,

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(a) Evaluate the robustness $\widehat{\alpha}$ and the opportunity $\widehat{\beta}$.

(b) Compare these two immunities by expressing one as a function of the other. Also, note their different variation with the threshold values, $r_{\rm c}$ and $r_{\rm w}$.

(c) Explain why "bigger is better" for $\widehat{\alpha}$, while "big is bad" for $\widehat{\beta}$.

Uniform bound:

$$\mathcal{U}(\alpha, \widetilde{u}) = \{u(t): \ |u(t) - \widetilde{u}(t)| \le \alpha\}, \quad \alpha \ge 0 \tag{3}$$

Energy bound:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{u(t): \ \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, {\rm d}t \le \alpha^2\right\}, \quad \alpha \ge 0 \tag{4}$$

Fourier ellipsoid bound. The uncertain function is expanded in a truncated Fourier series:

$$u(t) = \widetilde{u}(t) + \sum_{n=n_1}^{n_2} [a_n \cos n\pi t + b_n \sin n\pi t] \tag{5}$$

$$= \widetilde{u}(t) + c^T \phi(t) \tag{6}$$

where c is the vector of uncertain Fourier coefficients and $\phi(t)$ is the vector of corresponding trigonometric functions. The info-gap model is:

$$\mathcal{U}(\alpha, 0) = \left\{u(t) = \widetilde{u}(t) + c^T \phi(t): \ c^T W c \le \alpha^2\right\}, \quad \alpha \ge 0 \tag{7}$$

where W is a known, real, symmetric, positive definite matrix.

2. **Robustness, opportunity and reward.** Consider an uncertain scalar u for which the reward function is:

$$R(q, u) = q_1 u + q_2 \tag{8}$$

where the coefficients q_1 and q_2 can be controlled by the decision maker, so that the decision vector is $q = (q_1, q_2)^T$. Small values of $R(q, u)$ are more desirable than large values.

For each of the info-gap models listed in problem 1,

(a) Evaluate the robustness and opportunity functions, $\widehat{\alpha}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$.

(b) In each case, explore the variation of these immunity functions as the decision vector q is changed. If q is modified to improve the robustness, does the opportunity improve or deteriorate? Do optima exist for the immunity functions? If not, impose constraints on q to allow an optimum. Do the robust-optimal and opportunity-optimal decisions agree?

3. **Linear optimization on an ellipsoid.** Let x be a vector in the N-dimensional ellipsoidal set:

$$\mathcal{X} = \left\{x: \ x^T W x \le 1\right\} \tag{9}$$

where W is a real, symmetric, positive definite matrix.

Let y be an N-vector of unit length:

$$y^T y = 1 \tag{10}$$

Find the vector y for which $x^T y$ is a maximum for all x in $\mathcal{X}$. We can state this more explicitly as follows. Define the function:

$$f(y) = \max_{x \in \mathcal{X}} x^T y \tag{11}$$

What we are seeking is the vector $\widehat{y}$ for which $f(y)$ is a maximum:

$$f(\widehat{y}) = \max_{y^T y = 1} f(y) \tag{12}$$

4. **Static deflection of a cantilever.** In many problems the reward function can be expressed as a linear function of an info-gap uncertain vector. We have seen some examples already. Here is another example.

   Consider a uniform cantilever beam subject to N static point loads applied perpendicular to the beam axis. The loads all lie in a single plane. The vector of loads is denoted:

$$f = (f_1, \ldots, f_N)^T \tag{13}$$

   The deflection of the free end of the cantilever, y, is linearly related to the loads by:

$$y = k^T f \tag{14}$$

   where k is a column vector of known flexibility coefficients.

   The load vector is uncertain and belongs to the following Fourier ellipsoid bound info-gap model:

$$F(\alpha, \widetilde{f}) = \left\{ f : \ \left(f - \widetilde{f}\right)^T W \left(f - \widetilde{f}\right) \leq \alpha^2 \right\}, \quad \alpha \geq 0 \tag{15}$$

   where W is a real, symmetric, positive definite matrix.

   (a) If α is given, find the load vector f which results in the maximum end deflection.

   (b) If α is given, find the maximum end deflection.

   (c) The beam fails if the end deflection exceeds the critical value $y_{\rm c}$. What is the robustness function of the beam?

   (d) Now consider the choice between two designs with flexibility vectors k_1 and k_2 for which:

$$k_1^T \widetilde{f} \ > \ k_2^T \widetilde{f} \tag{16}$$
$$k_1^T W^{-1} k_1 \ < \ k_2^T W^{-1} k_2 \tag{17}$$

   (e) Now assume that α is a random variable with a known exponential distribution. Combine this information with the info-gap uncertainty about the load vector f to evaluate a hybrid robustness.

5. **Dynamic deflection of a cantilever.** Consider a rigid cantilever of length L and mass μ. The angle $\theta(t)$ between the cantilever axis and the support oscillates with linear rotational stiffness k [Nm/radian]. An external moment of force $M(t)$ is applied to the free end of the cantilever. For the initial T seconds of oscillation the moment of force varies from the nominal value M_o in an unknown but bounded manner; after time T the moment vanishes. We can represent the uncertainty in $M(t)$ with the envelope-bound info-gap model:

$$\mathcal{U}(\alpha, M_o) = \{ M(t) : \ |M(t) - M_o| \leq \alpha \psi(t) \}, \quad \alpha \geq 0 \tag{18}$$

   with the envelope function $\psi(t)$ chosen as:

$$\psi(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & t \geq T \end{cases} \tag{19}$$

   The angular deflection $\theta(t)$ of the rigid beam is described by:

$$J\ddot{\theta}(t) + k\theta(t) = M(t) \tag{20}$$

   where the moment of inertia of the beam is $J = \mu L^2/3$.

   The system can tolerate deviation of the angular deflection of the beam by no more than $\theta_{\rm c}$. Ideally, the angular deflection is less than a much smaller threshold, $\theta_{\rm w}$.

(a) Evaluate the robustness of the system, in terms of the amount of input uncertainty which the system can tolerate without failing. Consider durations less than T and much longer than T.

(b) Evaluate the opportunity of the system, in terms of the amount of input uncertainty which must be present in order for it to be possible that the beam deviates by no more than $\theta_{\rm w}$. Consider durations less than T and much longer than T.

(c) Consider the stiffness k and the moment of inertia J as decision variables, so $q = (k, J)^T$ is the decision vector. Explore the variation of the robustness and of the opportunity with q.

6. **Static deflection of a cantilever: continued.** In some problems the reward function can be expressed as a quadratic function of an info-gap uncertain vector. We modify problem 4 as an example. Instead of the deflection y in eq.(14), which is a linear performance function, let us consider:

$$r = f^T V f \tag{21}$$

r is proportional to an elastic energy of deformation. V is a known, real, symmetric, positive definite matrix.

Re-examine questions 4a–4c for the following two cases: **(a)** $\widetilde{f} = 0$. **(b)** $\widetilde{f} \neq 0$.

7. **Flaw-resistant manufacture.** An automatic milling machine is equipped with a sensor system which detects flaws in the workpiece. When a flaw is detected, the cutting tool is raised above the surface of the work piece. This lifting mechanism is functional provided that the cutting tool is raised at least 1mm above the surface within 2 seconds of detecting the flaw. This is because the height of the workpiece varies by less than 1mm during 2 seconds of feed. The height of the cutting tool as a function of time develops nominally according to the function:

$$\widetilde{x}(t) = ht^2 \quad [\text{mm}] \tag{22}$$

However, there is uncertainty in the response of the lifting mechanism due to wear in the power chain. In other words, the height as a function of time does not always behave according to $\widetilde{x}(t)$. Uncertainty in the height as a function of time is described by the uniform-bound info-gap model:

$$\mathcal{U}(\alpha, \widetilde{x}) = \{x(t) : \ |x(t) - \widetilde{x}(t)| \leq \alpha\}, \quad \alpha \geq 0 \tag{23}$$

The uncertainty parameter α describes the inaccuracy in the tool height as a result of wear in the power chain. What is the robustness of the lifting mechanism? What is the opportunity function? What is the interpretation of these immunity functions, and how are they used in evaluating the nominal performance of the flaw-recovery system?

8. **Quality control.** A long, thin, cylindrically symmetrical surgical needle is finely milled to conform to its specified shape, $\widetilde{t}(x)$. The allowed deviation of the actual thickness $t(x)$ from the designed thickness $\widetilde{t}(x)$ is $\pm D$ microns throughout the length L of the needle. The manufacturer guarantees the following two conditions:

(i) The precise dimension of the needle will be verified at each end by direct measurement. Any needle for which $t(0) \neq \widetilde{t}(0)$ or $t(L) \neq \widetilde{t}(L)$ will be rejected. This quality control measurement is completely accurate (or its accuracy is vastly greater than the allowed tolerance).

(ii) The deviation of the slope of the surface of the needle from its specified slope is bounded, in an attempt to exclude bristles and dents. However, the value of the bound on the slope is unknown.

(a) Formulate an info-gap model for the shape of needles produced to these specifications.

(b) Construct the robustness and opportunity functions for this manufacturing process.

(c) The manufacturer now proposes to replace the first condition above with the following extended quality check:

(i′) The precise dimension of the needle will be verified at each end *and* at N equi-distant intermediate points by direct measurement. Any needle which deviates from the specified dimension at any measured point will be rejected.

The second condition, (ii), remains valid. Construct the immunity functions and use them to choose the number N of measurement points. In particular, study the marginal utility of the Nth measurement.

(d) ‡ What modification of the initial information, and consequently of the info-gap model, would lead to a more meaningful opportunity function?

(e) ‡ Modify the quality control specification in item (i) to consider error in the quality control measurement. Specifically, any needle for which $|t(0) - \widetilde{t}(0)| > \varepsilon D$ or $|t(L) - \widetilde{t}(L)| > \varepsilon D$ will be rejected, where $0 \le \varepsilon \le 1$. Now repeat questions 8a and 8b for the robustness function only.

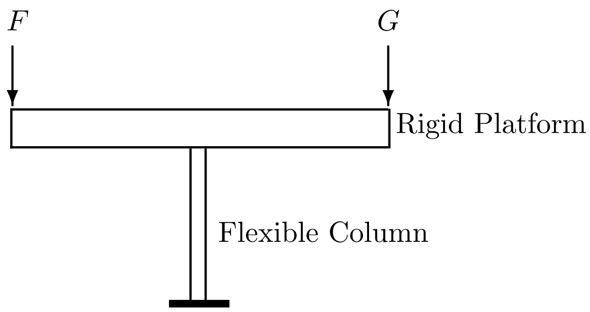

Figure 1: Platform for problem 9.

9. **Dynamic stability of a platform.** A rigid beam-like platform is supported from below at its midpoint by a flexible column which is at elastic equilibrium when the platform is horizontal, as shown in fig. 1. The flexural stiffness of the elastic column is k [Nm/radian] and it applies a restoring moment of force $M = k\theta$ when the platform is tilted by θ radians. The width of the platform is $2L$ [m]. The platform is loaded at its two ends by static forces F and G which are uncertain but bounded. That is, forces F and G belong to the following info-gap model of uncertainty:

$$\mathcal{U}(\alpha, 0) = \{F, G: \ |F| \ \le \ \alpha, \quad |G| \ \le \ \alpha\}, \quad \alpha \ge 0 \tag{24}$$

The platform is satisfactorily level if the angle of tilt at static equilibrium is never greater than the critical value θ_c:

$$|\theta| \le \theta_c \tag{25}$$

The condition of static equilibrium requires that the moment of force at the midpoint vanish:

$$0 = FL - GL + k\theta \tag{26}$$

Determine the robustness and opportunity functions of the platform. The decision vector is $q = (k, L)^T$. Study the variation of the immunity functions as these design variables are changed.

10. **Dynamic stability of a platform: continued.** We now modify problem 9 to consider uncertain distributed loads, $f(x)$ [N/m], $-L \le x \le L$, on the platform. Evaluate the robustness and opportunity for each of the following info-gap models for uncertainty in the load.

(a) *Uniform-bound:*

$$\mathcal{U}(\alpha, \widetilde{f}) = \left\{ f(x) : \ \left| f(x) - \widetilde{f}\cos\frac{\pi x}{L} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{27}$$

where $\widetilde{f}$ is a known constant.

(b) *Fourier ellipsoid bound:* The uncertain part of the load profile is a truncated sine series:

$$\begin{aligned} f(x) &= \widetilde{f}\cos\frac{\pi x}{L} + \sum_{n=1}^{N} c_n \sin\frac{n\pi x}{L} && (28) \\ &= \widetilde{f}\cos\frac{\pi x}{L} + c^T\sigma(x) && (29) \end{aligned}$$

where c is the vector of uncertain Fourier coefficients and $\sigma(x)$ is the vector of sine functions. The info-gap model is:

$$\mathcal{U}(\alpha, \widetilde{f}) = \left\{ f(x) = \widetilde{f}\cos\frac{\pi x}{L} + c^T\sigma(x) : \ c^T W c \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{30}$$

where W is a known, real, symmetric, positive definite matrix.

(c) *Different nominal load.* How will the answers to questions 10a and 10b change if the nominal load is:

$$\widetilde{f}(x) = \widetilde{f}\sin\frac{\pi x}{L} \tag{31}$$

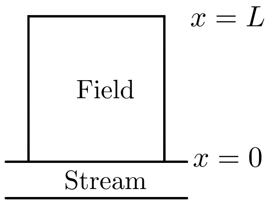

Figure 2: Illustration for problem 11.

11. **Environmental contamination.** The concentration of contaminant, $f(x)$, varies over a farmer's field as a function of distance x from a stream passing along one side of the field:

$$\begin{aligned} f(x) &= \widetilde{f} + \sum_{n=1}^{N} \phi_n x^n && (32) \\ &= \widetilde{f} + \phi^T\xi && (33) \end{aligned}$$

where ϕ is the vector of unknown coefficients and ξ is the corresponding vector of powers of x. The nominal concentration, $\widetilde{f}$, is a known constant. The stream is located at $x = 0$ and the far edge of the field is at $x = L$.

The quantity of contaminant which reaches the stream by the end of the season is:

$$g = \mu \int_0^L \left[1 - \left(\frac{x}{L} \right)^{\nu} \right] f(x)\,\mathrm{d}x \tag{34}$$

where μ and ν are positive constants which reflect absorption and transport properties of the soil, and which can be influenced in known ways by treating the field.

The water in the stream is potable if the quantity of contaminant does not exceed the value g_{c}, but it is highly desirable that the quantity of contaminant not exceed the far lower value g_{w}.

For the Fourier ellipsoid bound info-gap model specified below, determine the robustness and opportunity functions and study their behavior as a function of the decision variables μ and ν. Discuss the choice of these variables. Consider the antagonism and sympathy of the immunity functions, $\widehat{\alpha}$ and $\widehat{\beta}$.

$$\mathcal{U}(\alpha, \widetilde{f}) = \left\{ f(x) = \widetilde{f} + \phi^T \xi : \ \phi^T W \phi \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{35}$$

where W is a known, real, symmetric, positive definite matrix.

12. **Financial investment.** Consider an investment problem in which a vector q of investments in N different options results in a vector x of returns in M different commodities. Investments and returns are related by:

$$x = Aq \tag{36}$$

where the $M \times N$ matrix A is uncertain.

The returns are satisfactory if:

$$x_m \ge x_{m,\mathrm{c}}, \quad m = 1, \ldots, M \tag{37}$$

The investment results in windfall returns if:

$$x_m \ge x_{m,\mathrm{w}}, \quad m = 1, \ldots, M \tag{38}$$

where:

$$x_{m,\mathrm{w}} > x_{m,\mathrm{c}}, \quad m = 1, \ldots, M \tag{39}$$

Construct the robustness and opportunity functions for each of the info-gap models listed below. Discuss the use of these immunity functions in evaluating an investment.

Interval bound:

$$\mathcal{U}(\alpha, \widetilde{A}) = \left\{ A : \ \left| A_{mn} - \widetilde{A}_{mn} \right| \le \alpha, \quad m = 1, \ldots, M, \quad n = 1, \ldots, N \right\}, \quad \alpha \ge 0 \tag{40}$$

Row-wise ellipsoid bound: For any matrix R, let R_m be a row vector denoting the mth row of R.

$$\mathcal{U}(\alpha, \widetilde{A}) = \left\{ A = \widetilde{A} + R : \ R_m W R_m^T \le \alpha^2, \ m = 1, \ldots, M \right\}, \quad \alpha \ge 0 \tag{41}$$

where W is a known, real, symmetric, positive definite matrix.

13. **Satellite targeting.** **(a)** A satellite is launched from point L directly at a stationary target located a distance D away at point T. The satellite moves in a single plane, but the slope of the satellite trajectory, with respect to the line LT, varies in an unknown manner during flight. The satellite carries a payload of photographic devices, which are just barely effective if the satellite-target distance is no greater than r_c as the satellite passes the target. The payload is highly effective if the fly-by distance is as small as r_w. Evaluate the robustness and opportunity functions.

    (b) Now consider a slightly more complex situation. The satellite is initially launched at a slope s_0 which is lower than the slope of the line from L to T. The slope of the satellite trajectory deviates in flight from s_0 in an unknown manner. The mission fails if the satellite passes too far from or too near to the target. That is, failure occurs if the satellite-target distance is greater than $r_{\mathrm{c},2}$ or less than $r_{\mathrm{c},1}$ as the satellite passes the target. The mission is highly successful if the satellite passes as near as r_w from the target, but not too near, where $r_{\mathrm{c},1} < r_\mathrm{w} < r_{\mathrm{c},2}$. Evaluate the robustness and opportunity functions.

14. ‡ **Heat conduction.** Consider an unknown one-dimensional heat source, $h(x)$ [W/m], distributed along x between $+1$ and -1. The temperature distribution is $T(x)$ degrees K. The heat-source density function, $h(x)$, is uncertain and belongs to a info-gap model.

The differential equation for heat conduction is:

$$0 = \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} + \frac{h(x)}{k} \tag{42}$$

where k is the thermal conductivity, in units of W·m/K.

Safe operation requires that the central temperature be less than a critical value:

$$T(0) \le T_{\mathrm{c}} \tag{43}$$

We are able to control the surface temperatures, $T(\pm 1)$.

Consider the following two info-gap models.

Uniform bound:

$$\mathcal{U}(\alpha, \widetilde{h}) = \left\{ h(x) : \ \left| h(x) - \widetilde{h} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{44}$$

Fourier ellipsoid bound:

$$\mathcal{U}(\alpha, \widetilde{h}) = \left\{ h(x) = \widetilde{h} + c^T \gamma(x) : \ c^T W c \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{45}$$

where W is a known, real, symmetric, positive definite matrix and $\gamma(x)$ is the vector:

$$\gamma(x) = (\cos \pi x, \, \cos 2\pi x, \, \ldots, \, \cos N\pi x)^T \tag{46}$$

(a) Study the robustness and the opportunity as a function of surface temperature, for each of the above info-gap models of heat-source uncertainty. Discuss the meaning of these two immunity functions. Develop general expressions for the immunity functions and then consider the special case where W is the following diagonal matrix:

$$W = \mathrm{diag}\left(\frac{1}{n^2}, \ n = 1, \ldots, 6 \right) \tag{47}$$

(b) Now consider a specific numerical case. The material is steel, whose thermal conductivity is $k = 17.3$ [W·m/K]. The critical temperature is $T_{\mathrm{c}} = 400$ [K]. The nominal heat-source density is $\widetilde{h} = 250$ [W/m]. For each of the info-gap models, what range of surface temperature values are very reliable? Very unreliable? Compare the results for the two info-gap models.

15. **Simple harmonic oscillator.** The displacement $x(t)$ of an oscillator in simple harmonic motion is:

$$m\ddot{x}(t) + kx(t) = u(t) \tag{48}$$

where the mass m and stiffness k are known but the driving force $u(t)$ is uncertain. The energy of the oscillator is proportional to the square of the displacement. Failure occurs if the energy at a specified time T exceeds a critical value:

$$x^2(T) \geq E_{\rm c} \tag{49}$$

The uncertainty in $u(t)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$u(t) = \widetilde{u}(t) + \sum_{n=1}^{N} \phi_n \sin \frac{n\pi t}{T} \tag{50}$$

$$= \widetilde{u}(t) + \phi^T \sigma(t) \tag{51}$$

where ϕ is the vector of uncertain Fourier coefficients and $\sigma(t)$ is the vector of corresponding sine functions. The info-gap model is:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u(t) = \widetilde{u}(t) + \phi^T \sigma(t) : \ \phi^T W \phi \leq \alpha^2 \right\}, \quad \alpha \geq 0 \tag{52}$$

where W is a known, real, symmetric positive definite matrix.

Evaluate the robustness and opportuneness functions and illustrate their use in choosing the system parameters, m and k. Assume that the initial displacement and velocity of the oscillator are both zero.

16. ‡ **Two coupled harmonic oscillators.** Consider two equal masses and three identical linear springs connected in sequence between two rigid walls as in fig. 3.

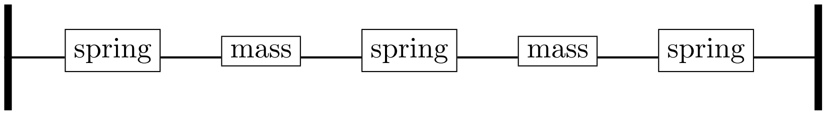

Figure 3: Mass-spring system for problem 16.

The masses are initially at rest in their equilibrium positions. An uncertain time varying force $u(t)$ is applied to the righthand mass. Let $x_1(t)$ and $x_2(t)$ denote the subsequent displacements of the left and righthand masses.

There are two natural modes of vibration of this system:

(a) In the 'coherent' mode the two masses oscillate together from right to left and back again. In this mode, the masses are always moving in the same direction.

(b) In the 'anti-coherent' mode the two masses are always moving in opposite directions: towards each other or away from each other.

Failure occurs if the amplitude of the incoherent mode exceeds the critical value $A_{\rm c}$.

Construct the robustness and opportunity functions of the system for each of the following info-gap models of uncertainty in the load:

Uniform bound:

$$\mathcal{U}(\alpha, 0) = \{u(t) : \ |u(t)| \le \alpha\}, \quad \alpha \ge 0 \tag{53}$$

Energy bound:

$$\mathcal{U}(\alpha, 0) = \left\{u(t) : \ \int_0^\infty u^2(t)\,{\rm d}t \le \alpha^2\right\}, \quad \alpha \ge 0 \tag{54}$$

Fourier ellipsoid bound:

$$u(t) = \sum_{n=1}^{N} c_n \sin \frac{n\pi t}{T} \tag{55}$$

$$= c^T \sigma(t) \tag{56}$$

where c is the vector of unknown Fourier coefficients and $\sigma(t)$ is the vector of corresponding sine functions.

$$\mathcal{U}(\alpha, 0) = \left\{u(t) = c^T \sigma(t) : \ c^T W c \le \alpha^2\right\}, \quad \alpha \ge 0 \tag{57}$$

where W is a known, real, symmetric, positive definite matrix.

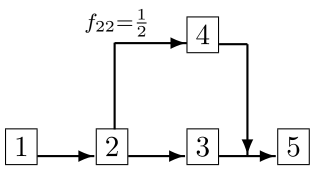

Figure 4: A 5-task project schedule for problem 17.

17. **Project management.** Consider the 5-task project shown in fig. 4. This project has two task-paths:

Path 1: $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5$
Path 2: $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5$

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = \widetilde{t}_2 = \widetilde{t}_5 = 1, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = 1 - q \tag{58}$$

where q is a parameter which the project manager is free to choose in the interval $[0, 1]$. q represents a valuable resource which must be allocated between tasks 3 and 4.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le \alpha, \ n = 1, \ldots, 5 \right\} \quad \alpha \ge 0 \tag{59}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.

18. **Project management: continued.** Consider a modification of problem 17. If the project completes in a duration T, then the "reward" is $R(T)$. $R(T)$ decreases as T grows to express the penalty associated with delayed termination. The project owner would very much like to earn reward as large as $r_{\rm w}$, and cannot tolerate reward less than $r_{\rm c}$. Formulate and evaluate the robustness and opportuneness functions. Indicate the choice of the parameter q. Explain how the robustness and opportuneness functions can be used to choose aspirations $r_{\rm c}$ and $r_{\rm w}$. Compare these results to the solution of problem 17.

19. ‡ **Project management: continued.** Consider the 16-task project in fig. 5.

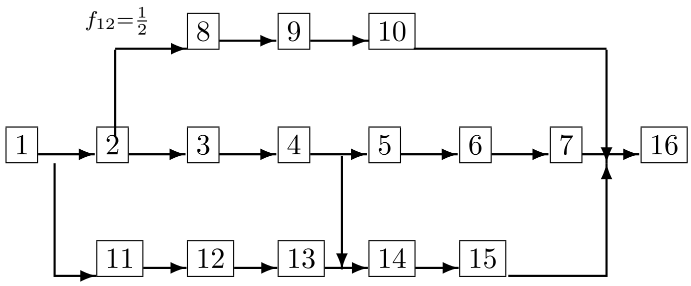

Figure 5: A 16-activity project schedule for problem 19.

t_n is the duration of the nth task, which is uncertain, and t is the vector of task durations. The project must be completed within the duration $T_{\rm c}$.

The info-gap model for uncertainty in t is:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le w_n \alpha, \ n = 1, \ldots, N \right\}, \quad \alpha \ge 0 \tag{60}$$

The nominal durations $\widetilde{t}_n$ and uncertainty weights w_n are given in table 1.

(a) Evaluate the dependence of the path-robustnesses and the overall robustness upon the participation factor f_{12} and the nominal duration $\widetilde{t}_2$ of task 2. Perform numerical calculations and discuss implications for improving the reliability of the project.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{t}_n$	1	1	2	3	3	3	2	1	2	3	3	3	1	3	2	1
w_n	1	1	1	1	1	1	1	1	1	1	3	2	2	3	2	1

Table 1: Nominal durations and uncertainty-weights for problem 19.

(b) Construct the opportunity function of the project and evaluate its dependence on the participation factor f_{12} and the nominal duration $\widetilde{t}_2$ of task 2. Discuss the meaning of this immunity function, and its use, in conjunction with the robustness function, in designing and managing the project.

(c) Evaluate the dependence of the path-robustnesses and the overall robustness upon the uncertainty weight w_{14} of task 14. Perform numerical calculations and discuss implications for improving the reliability of the project.

(d) We now modify the performance requirement of the project. Time over-runs are allowed but penalized, while early completion is rewarded. The reward function for completing the project in duration T is:

$$R(T) = R_0 \mathrm{e}^{-\mu(T-T_\mathrm{c})} \tag{61}$$

where T is the total project duration, and T_c, R_0 and μ are known positive constants.

The project manager is instructed to select an "attainable" reward goal r_c, and to choose a "realistic" project duration T_c.

Determine the robustness and opportunity functions for completion of the project. Study the variation of these immunity functions with T_c and r_c. Explain how these results assist the project manager in choosing T_c and r_c.

20. **Control of a production system.** The performance of a production system depends on temperature T, pressure P and time θ according to the relation:

$$y = T - 2P + \theta/3 \tag{62}$$

Define the vectors $x^T = (T,\ P,\ \theta)$ and $z^T = (1,\ -2,\ 1/3)$. Eq.(62) becomes:

$$y = z^T x \tag{63}$$

The system operator chooses operating values for T, P and θ, which will be denoted $\widetilde{T}$, $\widetilde{P}$ and $\widetilde{\theta}$, or collectively: $\widetilde{x}^T = (\widetilde{T},\ \widetilde{P},\ \widetilde{\theta})$.

The actual implemented value of x can vary in an unknown manner from the selected operational value $\widetilde{x}$. The uncertain deviation of x from $\widetilde{x}$ is described by an info-gap model:

$$\mathcal{U}(\alpha, \widetilde{x}) = \left\{ x :\ \left| \frac{x_i - \widetilde{x}_i}{\widetilde{x}_i} \right| \le \alpha, \quad i = 1,\ 2,\ 3 \right\}, \quad \alpha \ge 0 \tag{64}$$

The product of the system is unsatisfactory if:

$$y > y_{\text{c}} \tag{65}$$

where y_{c} is a known value.

(a) Derive an expression for the robustness of this production system, to uncertain variation of the system parameters.

(b) Consider the choice of the operational value $\widetilde{T}$, which must be selected in the interval $69^{\text{o}}\text{C} \le \widetilde{T} \le 85^{\text{o}}\text{C}$. What value would you recommend, and why?

21. **Reliability of a milling process.** An automated cutting tool moves at constant horizontal velocity across a work piece. The height $y(t)$ of the tool varies in transit. The desired height profile is $\widetilde{y}(t)$:

$$\widetilde{y}(t) = \sum_{n=n_1}^{n_2} \widetilde{b}_n \cos \frac{n\pi t}{T} = \widetilde{b}^T \gamma(t), \quad 0 \le t \le T \tag{66}$$

where $\widetilde{b}$ is the vector of known Fourier coefficients and $\gamma(t)$ is the vector of corresponding cosine functions.

The actual height profile differs from $\widetilde{y}(t)$ in an uncertain manner:

$$y(t) = \widetilde{y}(t) + \sum_{n=n_1}^{n_2} b_n \cos \frac{n\pi t}{T} = \widetilde{y}(t) + b^T \gamma(t), \quad 0 \le t \le T \tag{67}$$

where b is a vector of unknown Fourier coefficients. Uncertainty in b is represented by the following info-gap model:

$$\mathcal{U}(\alpha, \widetilde{b}) = \left\{ b : \left| \frac{b_n}{\widetilde{b}_n} \right| \le \alpha, \quad n = n_1, \ldots, n_2 \right\}, \quad \alpha \ge 0 \tag{68}$$

The milling process fails if the cutting tool is too far above the planned height at the end of the run, $t = T$. That is, failure is defined as:

$$y(T) - \widetilde{y}(T) > D_{\rm c} \tag{69}$$

(a) Formulate an expression which defines the robustness function.

(b) Derive an explicit algebraic expression for the greatest deviation of the actual height above the planned height, up to uncertainty α.

(c) Derive an explicit algebraic expression for the robustness function.

(d) Repeat parts (a)–(c) if the failure criterion is revised as follows. Failure occurs if the actual height deviates by more than $D_{\rm c}$ from the planned height at any time during the milling run. That is, failure is:

$$|y(t) - \widetilde{y}(t)| > D_{\rm c}, \quad 0 \le t \le T \tag{70}$$

22. **Reliability of robotic motion.** The arm of a robot moves in the (x, y) plane. The trajectory of the end effector as a function of time t is specified by:

$$x(t) = c_1(t+1) \tag{71}$$

$$y(t) = \frac{c_2}{t+1} \tag{72}$$

for $t \ge 0$. The coefficients c_1 and c_2 are uncertain and the uncertainty is described by the following info-gap model:

$$\mathcal{U}(\alpha, \widetilde{c}) = \{ c : \ |c_i - \widetilde{c}_i| \le \alpha \widetilde{c}_i, \quad i = 1, 2 \}, \quad \alpha \ge 0 \tag{73}$$

The values of $\widetilde{c}_i$ are known and positive.

The end effector must reach the following location at specified time $t = T$:

$$\widetilde{x} = \widetilde{c}_1(T+1) \tag{74}$$

$$\widetilde{y} = \frac{\widetilde{c}_2}{T+1} \tag{75}$$

The robot motion fails if the end-effector location at time T deviates from the desired location by more than a distance $D_{\rm c}$.

(a) Derive an expression for the robustness of the robotic motion.

(b) What is the optimal value of T, from the point of view of robustness? (Note that the final coordinates, $\widetilde{x}$ and $\widetilde{y}$, change as T changes. Thus, we aren't really concerned with *where* the end effector ends up, but just that it *meet* an object whose position will be $(\widetilde{x}, \widetilde{y})$.)

23. **Planning machine initialization.** A manufacturing machine runs continuously, and the longer it runs, the more it produces. However, the productivity, in terms of number of items produced per unit time, decreases the longer the machine runs. The total amount produced in time t is $g(t)$, which is an increasing function whose slope decreases in time. The function $g(t)$ is imperfectly known. An info-gap model for uncertainty in $g(t)$ is:

$$\mathcal{U}(\alpha, \widetilde{g}) = \{g(t) : \ |g(t) - \widetilde{g}(t)| \leq \alpha h(t)\}, \quad \alpha \geq 0 \tag{76}$$

where $\widetilde{g}(t)$ and $h(t)$ are known functions.

You must plan the production schedule for a total time of T hours. If the machine runs continuously for T hours its production will be $g(T)$. You can plan to stop and initialize the machine periodically. After each initialization the machine restarts with its initial productivity, which is high. However, each initialization requires τ hours. You must choose the number of restarts, assuming that the machine will run for the same duration after each initialization. The goal is to assure that the total production will not be less than r_{c}. The first initialization of the machine, like all subsequent initializations, requires τ hours.

(a) Derive an explicit expression for the robustness to uncertainty in the production function, when the machine is initialized n times during the T hours, assuming that the machine operates for the same duration after each initialization.

(b) Now assume that:

$$\widetilde{g}(t) = b\sqrt{t} \tag{77}$$

$$h(t) = \sqrt{t} \tag{78}$$

What is the robust-optimal number of initializations?

24. **Efficient fuel allocation.** The distance which a vehicle can travel with quantity q of fuel is:

$$f(q, c) = \frac{q}{1 + cq^2} \tag{79}$$

where c is an uncertain parameter described by the following info-gap model:

$$\mathcal{U}(\alpha, \widetilde{c}) = \{c : \ |c - \widetilde{c}| \leq \alpha \widetilde{c}\}, \quad \alpha \geq 0 \tag{80}$$

The vehicle must be able to travel at least a distance f_{c}.

(a) Find an explicit algebraic expression for the robustness of fuel quantity q to uncertainty in the coefficient c.

(b) Find an explicit algebraic expression for the quantity of fuel quantity q which maximizes the robustness. Compare this to the value of q which maximizes the distance based on the nominal value of c.

(c) Show that the robustness curves for different quantities of fuel can cross. What does this imply for the choice of fuel quantity?

25. **Moments on a robotic arm.** The angle of rotation of a robotic arm, y, varies according to the moments M_i applied at the joints according to:

$$y = \sum_{i=1}^{3} k_i M_i \tag{81}$$

where k_i is a known positive stiffness parameter. The moments M_i are poorly known. The best estimate of M_i is $\widetilde{M}_i$ which is positive. The fractional error of M_i is unknown:

$$\left| \frac{M_i - \widetilde{M}_i}{\widetilde{M}_i} \right| \leq \alpha, \quad \alpha \geq 0 \tag{82}$$

(a) The angle of rotation must be at least y_c. What is the robustness, to uncertainty in the moments, of the robotic rotation?

(b) In the solution to part (a) you found that the robustness depends on the nominal (anticipated) angle of rotation, $k^T \widetilde{M}$. By considering the robustness function, discuss whether it is desirable to design the robot so that $k^T \widetilde{M}$ is large or small.

(c) The nominal moments $\widetilde{M}_i$ can be chosen subject to the constraint:

$$\sum_{i=1}^{3} \widetilde{M}_i^2 = \mu^2 \tag{83}$$

where μ is known. What choice of the nominal moments maximizes the robustness?

26. **Machine efficiency.** The efficiency of a machine is described by:

$$h(q) = q + \frac{c}{q} \tag{84}$$

where $q > 0$ and c is uncertain and described by an info-gap model:

$$\mathcal{U}(\alpha, \widetilde{c}) = \{c: \ |c - \widetilde{c}| \le \alpha\sigma\}, \quad \alpha \ge 0 \tag{85}$$

(a) It is required that $h(q)$ be no less than h_c. What is the robustness of the machine to uncertainty in c, for a given value of q?

(b) The designer can choose q in the interval $[q_1, q_2]$ What choice of q do you recommend?

27. **Uncertain lotteries.** Consider a lottery with two prizes whose values are $v_\ell > v_s$. Each participant wins either the large prize or the small prize. The probability of winning the larger prize is uncertain; the best estimate of this probability if $\widetilde{p}$; and the info-gap model for $\widetilde{p}$ is:

$$\mathcal{U}(\alpha, \widetilde{p}) = \left\{p: \ 0 \le p \le 1, \ \left|\frac{p - \widetilde{p}}{\widetilde{p}}\right| \le \alpha\right\}, \quad \alpha \ge 0 \tag{86}$$

(a) For any critical reward v_c, such as the cost of a lottery ticket, what is the robustness, to uncertainty in $\widetilde{p}$, of winning at least v_c?

(b) Now consider a different lottery with prizes $v'_\ell > v'_s$ and estimated probability $\widetilde{p}'$ of winning v'_ℓ. Furthermore, the estimated average prize is now greater: $\widetilde{p}'v'_\ell + (1 - \widetilde{p}')v'_s > \widetilde{p}v_\ell + (1 - \widetilde{p})v_s$. However, the smaller prize is now even smaller: $v'_s < v_s$. The uncertainty $\widetilde{p}'$ is represented with the info-gap model of eq.(86). Under what conditions (e.g., with what values of v_c) will you prefer this new lottery? (Consider the crossing of the robustness curves of these two lotteries.)

28. **Braking system.** Consider a braking system upon which force $f(t)$ acts and for which the stopping distance is:

$$s(h, f) = \int_0^\infty h(t) f(t) \, \mathrm{d}t \tag{87}$$

The estimated braking function is:

$$\widetilde{h}(t) = \mathrm{e}^{-\mu t} \sin \omega t \tag{88}$$

The physics of braking and energy dissipation is complex and poorly understood. An info-gap model for uncertainty in the braking function is:

$$\mathcal{U}(\alpha, \widetilde{h}) = \left\{h(t): \ |h(t) - \widetilde{h}(t)| \le \alpha\right\}, \quad \alpha \ge 0 \tag{89}$$

(a) Given the driving function $f(t)$, and the requirement that the stopping distance not exceed s_c, derive the robustness function.

(b) We must choose between two driving functions, $f_1(t)$ and $f_2(t)$, where:

$$\int_0^\infty |f_1(t)| \, \mathrm{d}t < \int_0^\infty |f_2(t)| \, \mathrm{d}t, \quad s(\widetilde{h}, f_1) > s(\widetilde{h}, f_2) \tag{90}$$

Use the robustness function to specify the range of s_c-values for which each driving function is preferred.

29. **Ballistics.** A missile is designed to follow the trajectory:

$$\widetilde{h}(x) = \frac{\theta x (D - x)}{D} \tag{91}$$

where x is the horizontal distance from the launch site and $\widetilde{h}(x)$ is the height of the trajectory.

(a) The actual trajectory is uncertain due to wind and other disturbances, and is described by an info-gap model:

$$\mathcal{U}(\alpha, \widetilde{h}) = \left\{ h(x) : \ |h(x) - \widetilde{h}(x)| \le \alpha x \right\}, \quad \alpha \ge 0 \tag{92}$$

Operational considerations require that the strike distance of the missile (which happens when the height is zero) be no less than $D_{\rm c}$. What is the robustness to uncertainty in the trajectory?

(b) The info-gap model of eq.(92) allows unrealistically erratic trajectories. Re-do part (a) with this info-gap model:

$$\mathcal{U}(\alpha, \widetilde{h}) = \left\{ h(x) : \ h(0) = 0, \ |h'(x) - \widetilde{h}'(x)| \le \frac{\alpha x}{D} \right\}, \quad \alpha \ge 0 \tag{93}$$

where the prime implies differentiation. Do you expect the new robustness to be less or greater than the robustness in part (a)? Might the answer depend on the value of $D_{\rm c}$?

30. **Strain energy.** The strain energy E of a mechanical system is described by:

$$E = x^T x \tag{94}$$

where x is proportional to a vector of strains which result from a vector f of forces:

$$x = Vf \tag{95}$$

where V is a known matrix. The forces are constrained to:

$$f^T f \le \alpha^2 \tag{96}$$

What is the greatest value of α for which the strain energy is no larger than $E_{\rm c}$?

31. **Non-linear force-deflection relation.** Equilibrium of a 1-dimensional system is specified by:

$$xf = k + kf^2 \tag{97}$$

where x is the deflection, f is the force and k is a known positive constant. The system is safe if $x \le x_{\rm c}$. The nominal force is $\widetilde{f} = 1$, for which the system is safe. The uncertainty of the actual force is described by an info-gap model:

$$\mathcal{U}(\alpha, \widetilde{f}) = \left\{ f : \ f \ge 0, \ |f - \widetilde{f}| \le \alpha \right\}, \quad \alpha \ge 0 \tag{98}$$

Derive the robustness function.

32. **NEW! Fortifying a defensive line.** It is required to fortify a defensive line along the interval $0 \le x \le 1$. If an attack occurs at position x, then the losses are $L(x)$. The probability density function (pdf) for an attack at x is $p(x)$. The expected losses are:

$$\mathrm{E}(L|p) = \int_0^1 L(x) p(x) \, \mathrm{d}x \tag{99}$$

It is required that the expected losses not exceed the critical value $E_{\rm c}$:

$$\mathrm{E}(L|p) \le E_{\rm c} \tag{100}$$

The pdf is uncertain, and its estimated form is $\widetilde{p}(x)$. The info-gap model for the pdf is:

$$\mathcal{U}(\alpha, \widetilde{p}) = \left\{ p(x) : \ p(x) \ge 0, \ \int_0^1 p(x) \, \mathrm{d}x = 1, \ |p(x) - \widetilde{p}(x)| \le \alpha \widetilde{p}(x) \right\}, \quad \alpha \ge 0 \tag{101}$$

(a) Given $\widetilde{p}(x) = 2x$ and $L(x) = \lambda$ which is constant, develop an algebraic expression for the robustness function.

(b) Given $\widetilde{p}(x) = 1$ and $L(x) = x$, develop an algebraic expression for the robustness function.

(c) Now we change the story a bit. The loss-function is uncertain, its best estimate is $\widetilde{L}(x)$, and the info-gap model for $L(x)$ is:

$$\mathcal{U}(\alpha, \widetilde{L}) = \left\{ L(x) : \ |L(x) - \widetilde{L}(x)| \le \alpha \widetilde{L}(x) \right\}, \quad \alpha \ge 0 \tag{102}$$

where $\widetilde{L}(x) \ge 0$ but $L(x)$ can be both negative and positive, representing losses and gains.

The pdf is known and equals:

$$p(x) = \frac{\widetilde{L}(x)}{\int_0^1 \widetilde{L}(x)\, \mathrm{d}x} \tag{103}$$

The requirement in eq.(100) still holds. Develop an algebraic expression for the robustness function.

(d) In continuation to part (c), let us suppose that we can plan the estimated loss function, $\widetilde{L}(x)$, for instance, by planning the fortification. The total estimated loss is specified as Λ:

$$\int_0^1 \widetilde{L}(x)\, \mathrm{d}x = \Lambda \tag{104}$$

How should we choose the estimated loss function so as to maximize the robustness with expected loss satisficed at E_{c}?

‡**(e)** Now we return to the story in parts (a) and (b), but suppose that the pdf for the spatial distribution of attacks is chosen *strategically* by the adversary. Specifically, we think (though we don't know for sure) that the adversary knows the fortification, and hence knows $\widetilde{L}(x)$. Let us suppose that he chooses the pdf as to be large where $\widetilde{L}(x)$ is large. That is, the estimated pdf is:

$$\widetilde{p}(x) = \frac{\widetilde{L}(x)}{\int_0^1 \widetilde{L}(x)\, \mathrm{d}x} \tag{105}$$

However, the adversary may adopt a totally different strategy. The info-gap model for our uncertainty in the actual pdf is given by eq.(101). Now repeat parts (a) and (b) with this new $\widetilde{p}(x)$.

33. **NEW! Spatial monitoring, simple.** The density $\rho(x)$ of some highly undesirable material (e.g., toxin, invasive species, chemical impurity, etc.) varies along a transect from $x = 0$ to $x = L$. The true value of the total quantity is $r(\rho) = \int_0^1 \rho(x)\,\mathrm{d}x$. If r exceeds a very small critical amount, r_c, then remedial action will be taken. You will perform N measurements to verify that *none* of this material is present at positions $x_1, \ldots, x_N$. The density tends to be constant along the transect, but the actual slope of the density varies by an unknown amount along the transect. Given the measurements, the following info-gap model represents the spatial uncertainty in the true density function:

$$\mathcal{U}(\alpha) = \left\{\rho(x) : \ \rho(x_i) = 0, \ i = 1, \ldots, N, \ |\rho'(x)| \le \alpha\right\}, \quad \alpha \ge 0 \tag{106}$$

(a) Suppose you perform two measurements, one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

(b) Suppose you perform $N+1$ evenly spaced measurements, including one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

34. **NEW! Spatial monitoring.** The density $\rho(x)$ of some material of interest (e.g., rare plants, valuable minerals, chemical impurity, seismic faults, etc.) varies along a transect from $x = 0$ to $x = 1$. You will perform N measurements, obtaining the results $m_i = \rho(x_i)$, $i = 1, \ldots, N$. Your estimate of the mean density is $\overline{m} = (1/N)\sum_{i=1}^N m_i$. The true value of the average density is $\mu = \int_0^1 \rho(x)\,\mathrm{d}x$. The density tends to be constant, but the actual slope of the density varies by an unknown amount along the transect. Given the measurements, the following info-gap model represents the spatial uncertainty in the true density function:

$$\mathcal{U}(\alpha) = \left\{\rho(x) : \ \rho(x_i) = m_i, \ i = 1, \ldots, N, \ |\rho'(x)| \le \alpha\right\}, \quad \alpha \ge 0 \tag{107}$$

You require that the absolute difference between the estimate, $\overline{m}$, and the true value, μ, be no greater than ε.

(a) Suppose you perform a single measurement at the midpoint, $x_1 = 1/2$. Formulate and evaluate the robustness to spatial uncertainty.

(b) Suppose you perform two measurements, one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

35. **NEW! Investment for bio-diversity.** You will invest a quantity q of resources in order to increase the bio-diversity of a nature reserve. The number of new species which will thrive in the reserve after the investment is:

$$N(q, u) = u_1 q + u_2 q^2 \tag{108}$$

where the coefficients u_i are uncertain. The project is a failure if the number of new species is less than N_c.

(a) The best estimates of the coefficients u_i are $\widetilde{u}_i$, where $\widetilde{u}_1 > 0$ and $\widetilde{u}_2 < 0$. These estimates are highly uncertain and we have no further information other than that $u_1 \ge 0$ and $u_2 \le 0$. Use the fractional-error info-gap model:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u : \ \begin{array}{rcl} \max[0, \ (1-\alpha)\widetilde{u}_1] \le & u_1 & \le (1+\alpha)\widetilde{u}_1 \\ (1+\alpha)\widetilde{u}_2 \le & u_2 & \le \min[0, \ (1-\alpha)\widetilde{u}_2] \end{array} \right\}, \quad \alpha \ge 0 \tag{109}$$

Evaluate the robustness of investment q with requirement N_c. Discuss the significance of the possible crossing of the robustness curves.

(b) Now consider additional information. We have an estimate, σ_i, of the error of the estimated value $\widetilde{u}_i$. Now use the following modification of the info-gap model in eq.(109):

$$\mathcal{U}(\alpha, \widetilde{u}) = \Big\{ u: \ \max[0, \ \widetilde{u}_1 - \alpha\sigma_1] \leq \ u_1 \ \leq \widetilde{u}_1 + \alpha\sigma_1 \\ \widetilde{u}_2 - \alpha\sigma_2 \leq \ u_2 \ \leq \min[0, \ \widetilde{u}_2 + \alpha\sigma_2] \Big\}, \quad \alpha \geq 0 \tag{110}$$

Evaluate the robustness of investment q with requirement $N_{\rm c}$ and discuss the significance of the possible crossing of the robustness curves.

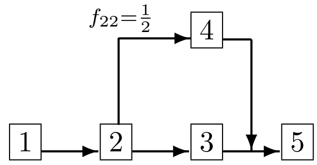

Figure 6: A 5-task project schedule for problem 17.

36. **Project management.** Consider the 5-task project shown in fig. 6. This project has two task-paths:

$$\begin{array}{ll} \text{Path 1:} & 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \\ \text{Path 2:} & 1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \end{array}$$

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = 1, \ \widetilde{t}_2 = 1.3, \ \widetilde{t}_5 = 1.5, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = 1 - q \tag{111}$$

where q is a parameter which the project manager is free to choose in the interval $[0, 1]$. q represents a valuable resource which must be allocated between tasks 3 and 4.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t: \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \leq \alpha, \ n = 1, \ldots, 5 \right\} \quad \alpha \geq 0 \tag{112}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.

37. **Project management.** Consider the 5-task project shown in fig. 4 on p.10. The modification from problem 17 is that we now consider both *allocation* between two tasks, and *total budget change.* This project has two task-paths:

$$\begin{array}{ll} \text{Path 1:} & 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \\ \text{Path 2:} & 1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \end{array}$$

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = \widetilde{t}_2 = \widetilde{t}_5 = 1, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = Q - q \tag{113}$$

where both Q (the total budget) and q (the allocation to one task) are parameters which the project manager is free to choose in the interval $[0, Q]$.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le \alpha, \ n = 1, \ldots, 5 \right\} \quad \alpha \ge 0 \tag{114}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.