Lecture Notes on

Info-Gap Uncertainties in Innovative Projects

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Source material:
• Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

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1 Managing Task-Duration Uncertainty

¶ Source material:

• Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London. Section 3.2.6, pp.64–70.

• Yakov Ben-Haim and Alexander Laufer, 1998, Robust reliability of projects with activity-duration uncertainty, *ASCE Journal of Construction Engineering and Management.* 124: 125–132.

• Avy Shtub, Sary Regev and Yakov Ben-Haim, 2000, Sikun m'hushav (Calculated Risk), *Tasiah v'Nihul,* issue #50, pp.32–37. In Hebrew.

¶ In this section we will consider a simple example of the info-gap analysis of task-duration uncertainty in a project. We will consider both robustness against failure and opportunity from uncertainty.

1.1 Basic Problem

¶ A project is characterized by:

- A flow-chart of tasks.
- Task-time estimates.
- Uncertainty in the duration of each task.
  (Alternatively: cost uncertainty.)
- Global requirement: complete project on time.
- Decision to be made: allocate a resource between two tasks.

¶ Questions:

- How robust is the project to task-duration uncertainty?
- How risky is the project?
- How can the robustness be increased (and the risk reduced)?
  - Re-structuring the project.
  - On-line monitoring.
  - Gathering information.
- How opportune is the project?
  Can windfalls be exploited?

1.2 Specific Problem Formulation

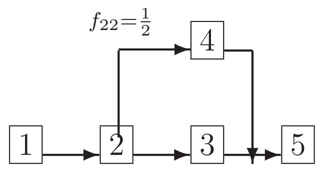


Figure 1: A 5-task project schedule for section 1.

¶ Consider the 5-task project shown in fig. 1. This project has two task-paths:

$$\begin{array}{ll} \text{Path 1:} & 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 5 \\ \text{Path 2:} & 1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \end{array}$$

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

¶ **Nominal task durations.**

- For tasks 1, 2 and 5:

$$\widetilde{t}_1 = \widetilde{t}_2 = \widetilde{t}_5 = 1, \tag{1}$$

- Estimated durations of tasks 3 and 4 depend on allocation of a resource:
  q_3 = resource allocated to task 3.
  q_4 = resource allocated to task 4.
  Budget constraint:

$$1 = q_3 + q_4 \tag{2}$$

- More resource decreases estimated task durations:

$$\widetilde{t}_3 = 1 - q_3 = q_4 \tag{3}$$
$$\widetilde{t}_4 = 1 - q_4 \tag{4}$$

Let $q = q_4$ be a parameter which the project manager is free to choose in the interval [0, 1].

¶ Decision: choose q.

This is based on analysis of robustness to uncertainty, which involves three components:

- An uncertainty model.
- A system model.
- A performance requirement.

¶ **Uncertainty model** for the task durations.
t_n = unknown true duration of nth task.
$\widetilde{t}_n$ = known estimate of duration of nth task.
Fractional-error info-gap model:

$$\mathcal{U}(\alpha, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le \alpha, \ n = 1, \ldots, 5 \right\} \quad \alpha \ge 0 \tag{5}$$

- $\dfrac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le \alpha$ is equivalent to:

$$\widetilde{t}_n - \alpha \widetilde{t}_n \le t_n \le \widetilde{t}_n + \alpha \widetilde{t}_n \tag{6}$$

- This is an unbounded family of nested sets.
- Two levels of uncertainty:
  - At fixed α: t_n, $n = 1, \ldots, N$ are uncertain.
  - α, the **uncertainty parameter,** is unknown.

¶ **System model and performance requirement:**
$t = (t_1, \ldots, t_N)$ = vector of task times.
$T(t)$ = total project duration = system model.
Requirement:

$$T(t) \le T_{\rm c} \tag{7}$$

1.3 Project Reliability with a Global Time Buffer: Theory

¶ Consider a project whose task flow chart is shown in fig. 1, p. 3.

¶ We first consider the **system model** in more detail.

¶ Participation matrix.
f_{mn} = fractional participation of task n in path m.
m: path. There are M paths.
n: task. There are N tasks.
In path m, the task following task n
begins when task n is fraction f_{mn} complete.

E.g., in path 2 of fig. 1 on p.3:
task 4 begins when task 2 is 1/2 complete: $f_{22} = 0.5$.

The full participation matrix is: For instance, for fig. 1:

$$F = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & 1 & 1 \end{pmatrix} \tag{8}$$

¶ The duration of the mth path, c_m,
equals the sum of the durations of **all tasks**
weighted by their fractional participations in path m:

$$c_m = \sum_{n=1}^{N} f_{mn} t_n, \quad m = 1, \ldots, M \tag{9}$$

For instance, the duration of the 2nd path is:

$$c_2 = 1 \cdot t_1 + \frac{1}{2} \cdot t_2 + 1 \cdot t_4 + 1 \cdot t_5 \tag{10}$$

¶ The **system model** is the duration of the longest path:

$$T(t) = \max_{1 \le m \le M} |c_m| = \max_{1 \le m \le M} \sum_{n=1}^{N} f_{mn} t_n \tag{11}$$

¶ **Robustness question:**

How wrong can our time-estimates be, and decision q still results in acceptable project duration?

¶ Robustness function:

$$\begin{aligned}
\widehat{\alpha} &= \max \alpha \text{ which precludes failure} && (12)\\
&= \max\{\alpha: \text{ failure is not possible}\} && (13)\\
&= \max\left\{\alpha: \ T \le t_{\rm c} \text{ for all } t \in \mathcal{U}(\alpha, \widetilde{t})\right\} && (14)\\
&= \max\left\{\alpha: \ \max_{1\le m\le M} \underbrace{\sum_{n=1}^{N} f_{mn} t_n}_{c_m} \le t_{\rm c} \text{ for all } t \in \mathcal{U}(\alpha, \widetilde{t})\right\} && (15)\\
&= \max\left\{\alpha: \ \max_{t\in\mathcal{U}(\alpha,\widetilde{t})} \max_{1\le m\le M} \sum_{n=1}^{N} f_{mn} t_n \le t_{\rm c}\right\} && (16)
\end{aligned}$$

¶ Analyze the maximum on t in eq.(16):

Recall that, for $t \in \mathcal{U}(\alpha, \widetilde{t})$:

$$\widetilde{t}_n - w_n \widetilde{t}_n \alpha \le t_n \le \widetilde{t}_n + w_n \widetilde{t}_n \alpha \tag{17}$$

Thus:

$$\begin{aligned}
\max_{t\in\mathcal{U}(\alpha,\widetilde{t})} c_m &= \max_{t\in\mathcal{U}(\alpha,\widetilde{t})} \sum_{n=1}^{N} f_{mn} t_n && (18)\\
&= \sum_{n=1}^{N} f_{mn}\left(\widetilde{t}_n + w_n \widetilde{t}_n \alpha\right) && (19)\\
&= \underbrace{\sum_{n=1}^{N} f_{mn} \widetilde{t}_n}_{\overline{c}_m} + \alpha \underbrace{\sum_{n=1}^{N} f_{mn} w_n \widetilde{t}_n}_{f_m} && (20)\\
&= \overline{c}_m + \alpha f_m && (21)
\end{aligned}$$

¶ The robustness is obtained by solving for α:

$$\max_{1\le m\le M} (\overline{c}_m + \alpha f_m) = t_{\rm c} \tag{22}$$

¶ We can decompose this according to the separate paths:

$$\widehat{\alpha}_m = \text{robustness of path } m \tag{23}$$

which is the solution for α of:

$$(\overline{c}_m + \alpha f_m) = t_{\rm c} \tag{24}$$

which is:

$$\widehat{\alpha}_m = \frac{t_{\rm c} - \overline{c}_m}{f_m} \tag{25}$$

for each $m = 1, \ldots, M$.

¶ One now sees that the project robustness is the lowest path-robustness:

$$\widehat{\alpha} = \min_{1 \le m \le M} \widehat{\alpha}_m \tag{26}$$

$$= \min_{1 \le m \le M} \frac{t_{\mathrm{c}} - \overline{c}_m}{f_m} \tag{27}$$

1.4 Robustness of the 5-Task Project

¶ First consider the **robustness function.** In this problem the uncertainty weights w_n are all equal to unity, so the robustness of task-path m, eq.(25), is:

$$\widehat{\alpha}_m = \frac{t_c - \widetilde{c}_m}{\widetilde{c}_m} \tag{28}$$

where

$$\widetilde{c}_m = \sum_{n=1}^{N} f_{mn}\widetilde{t}_n \tag{29}$$

The robustness is:

$$\widehat{\alpha} = \min_m \widehat{\alpha}_m \tag{30}$$

The participation matrix is:

$$F = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & 1 & 1 \end{pmatrix} \tag{31}$$

Hence:

$$\widetilde{c}_1 = \widetilde{t}_1 + \widetilde{t}_2 + q + \widetilde{t}_5 = 3 + q \tag{32}$$

$$\widetilde{c}_2 = \widetilde{t}_1 + \frac{1}{2}\widetilde{t}_2 + 1 - q + \widetilde{t}_5 = 3.5 - q \tag{33}$$

So the path robustnesses are:

$$\widehat{\alpha}_1(q) = \frac{t_c}{3+q} - 1 \tag{34}$$

$$\widehat{\alpha}_2(q) = \frac{t_c}{3.5-q} - 1 \tag{35}$$

¶ We see that $\widehat{\alpha}_1(q)$ decreases monotonically with q while $\widehat{\alpha}_2(q)$ increases monotonically with q, as shown in fig. 2. The overall robustness will be the lesser of these two functions at any value of q. That is, from this it is evident that the project robustness is:

$$\widehat{\alpha} = \begin{cases} \widehat{\alpha}_2(q) & , \quad 0 \le q \le 0.25 \\ \widehat{\alpha}_1(q) & , \quad 0.25 \le q \le 1 \end{cases} \tag{36}$$

Consequently we will choose q to maximize $\widehat{\alpha}$:

$$\widehat{q}_c = 0.25 \tag{37}$$

Fig. 3 shows the robustness, $\widehat{\alpha}(q, T_c)$, evaluated at the robust-optimal allocation, $q = 0.25$.

¶ Figs. 4 and 5 provide additional insight into why the robust-optimal allocation, $\widehat{q}_c$, occurs when the path robustnesses are equal. By increasing the robustness of the more vulnerable path one necessarily reduces the robustness of the more immune path. The maximum project robustness occurs when the two path robustnesses are equal.

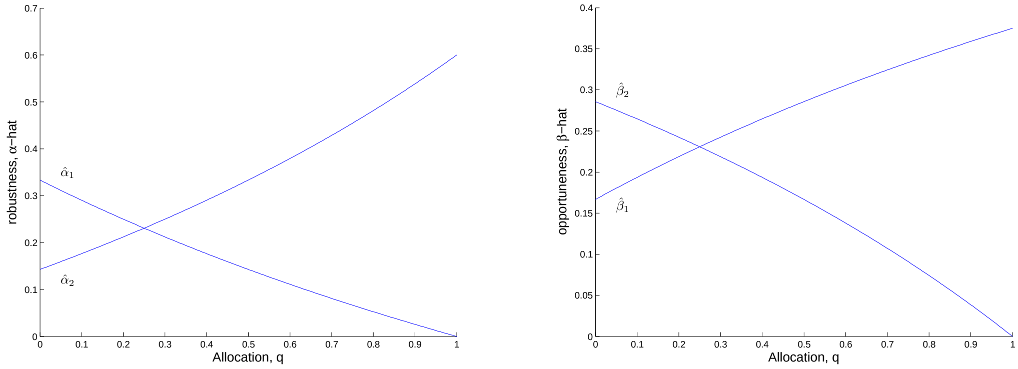


Figure 2: Path-robustness curves, eqs.(34) and (35), and opportuneness curves, eqs.(48) and (49). $T_\mathrm{c} = 4$, $t_\mathrm{w} = 2.5$.

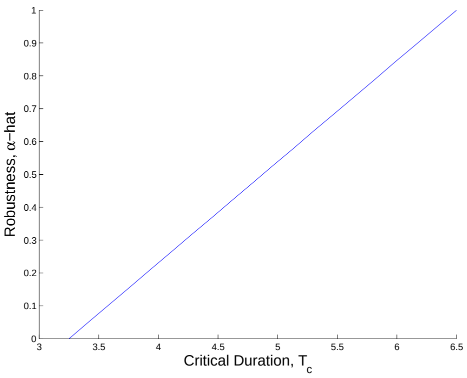


Figure 3: Project robustness curve at robust-optimal allocation, $q = 0.25$.

¶ Note that the path robustnesses shown in figs. 4 and 5 become zero for specific values of required completion time t_c. This means that the project robustness vanishes for some value of t_c. One can show that:

$$\widehat{\alpha}(q, t_\mathrm{c}) = 0 \quad \text{if} \quad t_\mathrm{c} = T[\widetilde{t}(q)] \tag{38}$$

$\widetilde{t}(q)$ is the vector of anticipated task-completion times, and $T[\widetilde{t}(q)]$ is the anticipated project completion time. What eq.(38) means is that one cannot rely upon attaining the anticipated completion time. Only longer, less ambitious, completion times have robustness to task-time uncertainty.

¶ Eq.(38) is true for *any* allocation q. In particular, eq.(38) holds for the allocation which minimizes the anticipated completion time:

$$q^\star = \arg\min_q T[\widetilde{t}(q)] \tag{39}$$

Allocation $q^\star$ is the "best" decision based on the "best" model, but its performance has zero robustness to uncertainty. Only "sub-optimal" allocations have positive robustness.

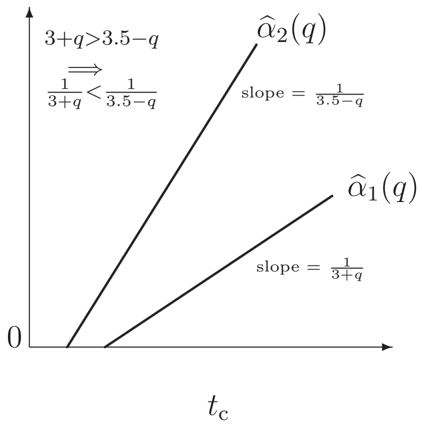


Figure 4: Path-robustness curves vs. completion time $t_{\rm c}$, eqs.(34) and (35).

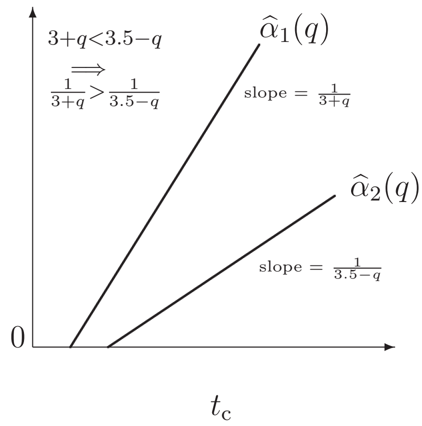


Figure 5: Path-robustness curves vs. completion time $t_{\rm c}$, eqs.(34) and (35).

1.5 Opportuneness Analysis of the 5-Task Project

¶ Now consider the **opportuneness function.**
¶ The opportuneness for task-path m is:

$$\widehat{\beta}_m = \min\left\{\alpha : \min_{t\in\mathcal{U}(\alpha,\widetilde{t})} c_m(t) \le t_{\rm w}\right\} \tag{40}$$

This is to be contrasted with the robustness of task-path m:

$$\widehat{\alpha}_m = \max\left\{\alpha : \max_{t\in\mathcal{U}(\alpha,\widetilde{t})} c_m(t) \le t_{\rm c}\right\} \tag{41}$$

¶ Can we use the info-gap model in eq.(5)? No. We must assure that task-durations are non-negative. Thus we need the following modification:

$$\mathcal{U}(\alpha,\widetilde{t}) = \left\{t : \max[0,\ \widetilde{t}_n(1-\alpha)] \le t_n \le \widetilde{t}_n(1+\alpha),\ n=1,\ldots,5\right\} \quad \alpha \ge 0 \tag{42}$$

¶ Note that:

$$\max[0,\ \widetilde{t}_n(1-\alpha)] = \begin{cases} 0\,, & \alpha \ge 1 \\ \widetilde{t}_n(1-\alpha)\,, & \alpha \le 1 \end{cases} \tag{43}$$

So it is evident that the opportuneness function will be less than $\widehat{\beta} = 1$, since at uncertainty $\alpha = 1$ each task *could* complete in zero time.

¶ Calculating minimal path duration at horizon of uncertainty α:

$$\min_{t\in\mathcal{U}(\alpha,\widetilde{t})} c_m(t) = \min_{t\in\mathcal{U}(\alpha,\widetilde{t})} \sum_{n=1}^{N} f_{mn} t_n \tag{44}$$

$$= \sum_{n=1}^{N} f_{mn}\widetilde{t}_n(1-\alpha) \tag{45}$$

$$= \widetilde{c}_m - \alpha\widetilde{c}_m \tag{46}$$

Equating this to $t_{\rm w}$ we see that the opportunity of path m is:

$$\widehat{\beta}_m = \frac{\widetilde{c}_m - t_{\rm w}}{\widetilde{c}_m} = 1 - \frac{t_{\rm w}}{\widetilde{c}_m} \tag{47}$$

unless this is negative, in which case $\widehat{\beta} = 0$. Explanation: eq.(47) is negative if $\widetilde{c}_m - t_{\rm w} < 0$ which means that the nominal duration is less than the windfall duration. In that case, the immunity to windfall is zero.

¶ The path opportunenesses are:

$$\widehat{\beta}_1(q) = 1 - \frac{t_{\rm w}}{3+q} \tag{48}$$

$$\widehat{\beta}_2(q) = 1 - \frac{t_{\rm w}}{3.5-q} \tag{49}$$

The overall opportuneness of the project is:

$$\widehat{\beta} = \max_m \widehat{\beta}_m \tag{50}$$

¶ We see that $\widehat{\beta}_1(q)$ increases monotonically with q while $\widehat{\beta}_2(q)$ decreases monotonically with q as shown in the righthand frame of fig. 2. The overall opportunity will be the greater of these two functions at any value of q. That is, from this it is evident that the project opportunity is:

$$\widehat{\beta} = \begin{cases} \widehat{\beta}_2(q) & , \quad 0 \le q \le 0.25 \\ \widehat{\beta}_1(q) & , \quad 0.25 \le q \le 1 \end{cases} \tag{51}$$

Consequently we will choose q to minimize $\widehat{\beta}$ which, as for the robustness function, leads to the same optimum:

$$\widehat{q}_{\rm w} = 0.25 \tag{52}$$

¶ Table 1 shows some $\widehat{\alpha}$ and $\widehat{\beta}$ values. Recall that $\widehat{\alpha}$ is the greatest fractional error in task-duration which the project can tolerate, in each task, without time over-run, while $\widehat{\beta}$ is the smallest 'error' (early task completion) which the project needs, in each task, to enable windfall project completion.

• Note that, in table 1 and (schematically) in fig. 6, $\widehat{\beta}(q) \le \widehat{\alpha}(q)$ for all $q \le 0.25$ when $t_{\rm w} = 3$. This means that the system can tolerate uncertainty at a level which facilitates windfall.

• However, in table 1 and (schematically) in fig. 7, the robustness and opportuneness curves cross at $q = 0.15$ when $t_{\rm w} = 2.7$. That is, $\widehat{\beta}(q) > \widehat{\alpha}(q)$ for $q < 0.15$. This means that, at $q < 0.15$, the project cannot tolerate the uncertainty needed to facilitate windfall.

q	$\widehat{\alpha} = \widehat{\alpha}_2$ $t_{\rm c} = 4$	$\widehat{\beta} = \widehat{\beta}_2$ $t_{\rm w} = 3$	$\widehat{\beta} = \widehat{\beta}_2$ $t_{\rm w} = 2.7$
0.00	0.14	0.14	0.23
0.05	0.16	0.13	0.22
0.15	0.19	0.10	0.19
0.25	0.23	0.077	0.17

Table 1: Robustness and opportunity for various resource allocations.

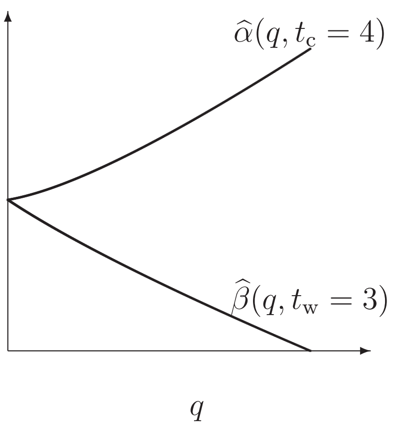


Figure 6: Robustness and opportuneness curves, with $t_{\rm c} = 4$ and $t_{\rm w} = 3$.

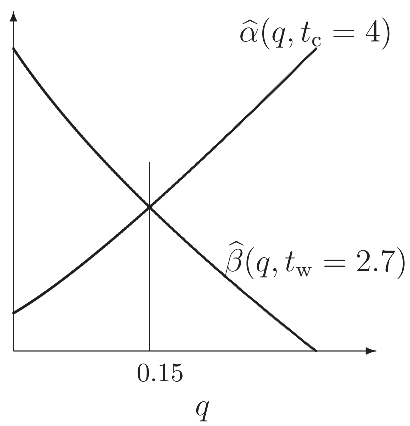


Figure 7: Robustness and opportuneness curves, with $t_{\rm c} = 4$ and $t_{\rm w} = 2.7$.

2 Project Cost Management

¶ **What are the info-gaps in cost management?**
- Unanticipated resource needs.
- Unanticipated resource costs.
- Both needs and costs are projections from the past to the future.
- Unanticipated budget constraints due to changing corporate plans and capabilities.

¶ **Costs:**
- c_i = true, unknown cost of ith task. $c = (c_1, \ldots, c_N)$.
- $\widetilde{c}_i$ = estimated cost of ith task. $\widetilde{c} = (\widetilde{c}_1, \ldots, \widetilde{c}_N)$.

¶ **Three components** of info-gap robustness analysis:
- Uncertainty model.
- System model: total discounted cost.
- Performance requirement: budget constraint.

¶ **Uncertainty model:** fractional-error info-gap model:

$$\mathcal{U}(\alpha, \widetilde{c}) = \left\{ c : \left| \frac{c_i - \widetilde{c}_i}{\widetilde{c}_i} \right| \leq \alpha w_i, \quad i = 1, \ldots, N \right\}, \quad \alpha \geq 0 \tag{53}$$

- Each cost varies in an expanding interval of unknown size:

$$(1 - w_i \alpha)\widetilde{c}_i \leq c_i \leq (1 + w_i \alpha)\widetilde{c}_i \tag{54}$$

- $\mathcal{U}(\alpha, \widetilde{c})$, $\alpha \geq 0$, is an unbounded family of nested sets of costs.
- $\mathcal{U}(\alpha, \widetilde{c})$ is analog of fractional-error info-gap model, eq.(5), on p.4.

¶ **System model:** total discounted cost:

$$C(c) = \sum_{i=1}^{N} d_i c_i \tag{55}$$

d_i = discount factor depending on time of purchase and interest rate.

¶ **Performance requirement:** budget constraint:

$$C(c) \leq B \tag{56}$$

¶ **Robustness question:** how wrong can our cost-estimate be, and we still stay within the budget?

¶ **Robustness:** greatest tolerable cost-estimate error:

$$\widehat{\alpha}(\widetilde{c}, d, B) = \max\left\{\alpha : \left(\max_{c \in \mathcal{U}(\alpha, \widetilde{c})} C(c)\right) \le B\right\} \tag{57}$$

¶ **Evaluating the robustness:**

- Inner maximum in eq.(57) occurs at maximum cost over-run:

$$\max_{c \in \mathcal{U}(\alpha, \widetilde{c})} C(c) = \sum_{i=1}^{N} d_i(1 + \alpha w_i)\widetilde{c}_i \tag{58}$$

- Equate this to B and solve for α to get the robustness:

$$\widehat{\alpha}(\widetilde{c}, d, B) = \frac{B - \sum_i d_i \widetilde{c}_i}{\sum_i d_i w_i \widetilde{c}_i} \tag{59}$$

- The estimated project cost is:

$$C(\widetilde{c}) = \sum_i d_i \widetilde{c}_i \tag{60}$$

- The "uncertainty-weighted" estimated project cost is:

$$W(\widetilde{c}) = \sum_i d_i w_i \widetilde{c}_i \tag{61}$$

¶ We can use eq.(59) to study several implications:
- Trade-off between robustness and budget leanness.
- The robustness-value of gathering information.
- Preference-reversal between alternative financing structures.

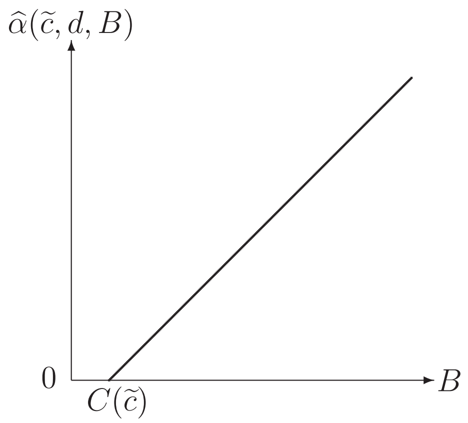


Figure 8: Trade-off between robustness and budget leanness.

¶ **Trade-off between robustness and budget leanness,** fig. 8.
- $C(\tilde{c})$ = estimated cost.
- $B - C(\tilde{c})$ = budget excess.
- Robustness increases as budget excess increases.
- Zero robustness when budget equals estimated cost.

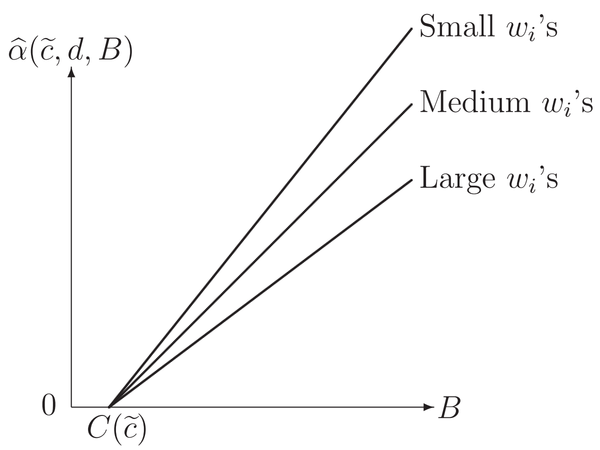


Figure 9: The robustness-value of gathering information.

¶ **The robustness-value of gathering information,** fig. 9.
- Gathering information reduces the w_i's.
- From eq.(59) we see that this increases the robustness.

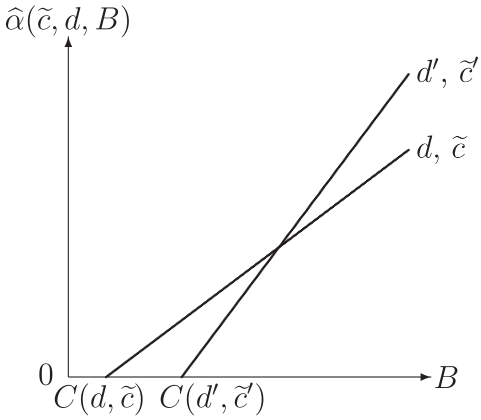


Figure 10: Preference-reversal between alternative financing structures.

¶ **Preference-reversal between alternative financing structures.**

- Two financing structures: $(d, \widetilde{c})$ and $(d', \widetilde{c}')$.
- The estimated cost of $(d', \widetilde{c}')$ exceeds the estimated cost of $(d, \widetilde{c})$:

$$\sum_i d_i \widetilde{c}_i < \sum_i d'_i \widetilde{c}'_i \tag{62}$$

- However, the uncertainty weights, w_i, are such that:

$$\sum_i d_i w_i \widetilde{c}_i > \sum_i d'_i w_i \widetilde{c}'_i \tag{63}$$

- This implies that the robustness curves cross, as in fig. 10.
- Thus the more expensive finance scheme, $(d', \widetilde{c}')$, is preferred if:
  - ○ Large robustness is needed, and,
  - ○ Large budget excess is available.

3 Expected Utility with Info-Gaps

3.1 The Problem

¶ **The problem:**
- Our firm is developing an innovative new product.
- We are expecting high value from marketing this product.
- However, we face uncertain competition, and hence uncertain market share.
- We must choose between alternative development options.

¶ In this lecture:
- We first develop the approach of **expected utility.**
- We then embed the expected utility analysis in an **info-gap robust-satisficing** approach.
- We will deal with three basic entities:
  - States of the world.
  - Actions.
  - Utilities.

1. **States of the world**, where p_j = probability that the world is in state j.

   The states of the world refer to the alternative possible market shares which our firm will take when the product reaches the market:

   (a) *No competitive or substitute products* will be available when our product reaches the market. We will have a monopoly market share. $j = 1$.

   (b) There will be *minor competitors* when we reach the market. We will have a dominant market share. $j = 2$.

   (c) There will be *major competitors* when we reach the market. We will have a minority, but significant, market share. $j = 3$.

2. **Actions,** denoted a_i, are alternative development options. These include:

   (a) Develop a high-cost and high-quality product which will attract upper-end customers. $i = 1$.

   (b) Develop a low-cost and moderate-quality product which will attract middle- and upper-end customers. $i = 2$.

3. **Utilities,** v_{ij} is the net value of action a_i if the world is in state j.

3.2 Expected Utility

¶ The **expected utility** of action a_i is the net value of that action, averaged over all states of the world:

$$\mathrm{E}(a_i) = \sum_j v_{ij} p_j \tag{64}$$

¶ The **optimal action,** $a^\star$, from the perspective of expected utility theory, is the action which maximizes the average net value:

$$a^\star = \arg\max_{a_i} \mathrm{E}(a_i) \tag{65}$$

$$= \arg\max_{a_i} \sum_j v_{ij} p_j \tag{66}$$

$a^\star$ is the action which, on average, has the highest utility (greatest net value), based on the values of v_{ij} and p_j in eq.(66).

3.3 Uncertainties

¶ The expected utility approach is designed to deal with:
- Uncertainty in the state of the world. Hence, the terms p_j.
- Net value from action a_i in state j, hence the utilities v_{ij}.

¶ However, the probabilities, p_j and the net values v_{ij}, are themselves very imprecisely known. There are large **info-gaps** between the best estimates and the true values of these quantities.

¶ We will represent the uncertainties in p_j and v_{ij} by info-gap models $\mathcal{P}(\alpha, \widetilde{p})$ and $\mathcal{V}(\alpha, \widetilde{v})$, respectively:

$$\mathcal{P}(\alpha, \widetilde{p}) = \left\{ p : \sum_j p_j = 1. \ \max[0, \ (1-\alpha)\widetilde{p}_j] \ \le \ p_j \ \le \ \min[1, \ (1+\alpha)\widetilde{p}_j], \ j = 1, \ 2, \ldots \right\}, \quad \alpha \ge 0 \tag{67}$$

- This is a fractional-error info-gap model.
- Since the p_j's are probabilities, they must lie in the interval [0, 1].
- The probability distribution p_j is normalized on j.

$$\mathcal{V}(\alpha, \widetilde{v}) = \{ v : \ \widetilde{v}_{ij} - \alpha w_{ij\mathrm{s}} \ \le \ v_{ij} \ \le \ \widetilde{v}_{ij} + \alpha w_{ij\ell}, \ \ i = 1, \ 2, \ldots j = 1, \ 2, \ldots \}, \quad \alpha \ge 0 \tag{68}$$

- This is an asymmetric-interval-bound info-gap model, in which the uncertainty weights $w_{ij\mathrm{s}}$ and $w_{ij\ell}$ are known and calculated as follows.
- Define:
  $\widetilde{v}_{ij\mathrm{s}}$ = estimated smallest net value from action i in state j.
  $\widetilde{v}_{ij\ell}$ = estimated largest net value from action i in state j.
- Now calculate:
  $w_{ij\mathrm{s}} = \widetilde{v}_{ij} - \widetilde{v}_{ij\mathrm{s}}$.
  $w_{ij\ell} = \widetilde{v}_{ij\ell} - \widetilde{v}_{ij}$.

3.4 Robustness

3.4.1 Formulation

¶ Given estimates $\widetilde{p}$ and $\widetilde{v}$ of the probabilities and utilities, we can estimate the expected utility of any action a_i, $\mathrm{E}(a_i, \widetilde{p}, \widetilde{v})$.

¶ For any other choice of the probabilities and utilities, p and v, the expected utility is $\mathrm{E}(a_i, p, v)$.

¶ Since these estimates, $\widetilde{p}$ and $\widetilde{v}$, are very uncertain, we do not have confidence that the actual utility which is expected to result from action a_i equals $\mathrm{E}(a_i, \widetilde{p}, \widetilde{v})$.

That is, we have every reason to believe that, for many choices of p and v, and especially for the true choice:

$$\mathrm{E}(a_i, p, v) \neq \mathrm{E}(a_i, \widetilde{p}, \widetilde{v}) \tag{69}$$

¶ Let E_{c} be the lowest level of expected utility (least average net value) which we are willing to accept.

¶ The robustness of action a_i, to uncertainties in p_j and v_{ij}, is the greatest horizon of uncertainty α up to which adequate expected utility, E_{c}, is obtained for any realization of p_j and v_{ij}:

$$\widehat{\alpha}(a_i, E_{\mathrm{c}}) = \max \left\{ \alpha : \left(\min_{\substack{p \in \mathcal{P}(\alpha, \widetilde{p}) \\ v \in \mathcal{V}(\alpha, \widetilde{v})}} \sum_{j=1}^{3} v_{ij} p_j \right) \geq E_{\mathrm{c}} \right\} \tag{70}$$

¶ The robust-satisficing action, $\widehat{a}(E_{\mathrm{c}})$, maximizes the robustness and satisfices the expected utility at the value E_{c}:

$$\widehat{a}(E_{\mathrm{c}}) = \arg \max_{a_i} \widehat{\alpha}(a_i, E_{\mathrm{c}}) \tag{71}$$

- $\widehat{a}(E_{\mathrm{c}})$, depends on the aspiration for average net value, E_{c}.
- $\widehat{a}(E_{\mathrm{c}})$, is likely to differ from $a^{\star}$, the action which maximizes the best-estimate of the average net value.

3.4.2 Evaluation

¶ In this section we explain how to evaluate the robustness, defined in eq.(70).

¶ First, we define an *indicator function* which will be useful:

$$h(x) \;=\; \begin{cases} 0, & \text{if} \quad x < 0 \\ x, & \text{if} \quad 0 \le x \le 1 \\ 1, & \text{if} \quad 1 < x \end{cases} \tag{72}$$

¶ Let us denote the inner minimum in eq.(70) by $\mu(\alpha)$.
- The robustness is the greatest value of α at which $\mu(\alpha) = E_{\rm c}$.
- Hence $\mu(\alpha)$ is the inverse of the robustness function, $\widehat{\alpha}(\text{Ec})$:
  A plot of $\mu(\alpha)$ vs. α is the same as a plot of $E_{\rm c}$ vs. $\widehat{\alpha}(\text{Ec})$.
- We will show how to calculate $\mu(\alpha)$.

¶ $\mu(\alpha)$ is obtained when the net values, v_{ij}, are as small as possible at horizon of uncertainty α:

$$v_{ij} = \widetilde{v}_{ij} - \alpha w_{ij{\rm s}} \tag{73}$$

¶ The choice of the probabilities, p_j, is more difficult.
- If $v_{ij} > 0$ then p_j should be minimal.
- If $v_{ij} < 0$ then p_j should be maximal.
- The probabilities, p_1, p_2, p_3, must be non-negative and normalized.
- Define $\text{sgn}(x)$ as the algebraic sign of x.
- There are three possibilities:

$$\begin{array}{llll} k=1: & p_1 = h[(1-\text{sgn}(v_{i1})\alpha)\widetilde{p}_1], & p_2 = h[(1-\text{sgn}(v_{i2})\alpha)\widetilde{p}_2], & p_3 = 1 - p_1 - p_2 \\ k=2: & p_1 = h[(1-\text{sgn}(v_{i1})\alpha)\widetilde{p}_1], & p_2 = 1 - p_1 - p_3, & p_3 = h[(1-\text{sgn}(v_{i3})\alpha)\widetilde{p}_3] \\ k=3: & p_1 = 1 - p_2 - p_3, & p_2 = h[(1-\text{sgn}(v_{i2})\alpha)\widetilde{p}_2], & p_3 = h[(1-\text{sgn}(v_{i3})\alpha)\widetilde{p}_3] \end{array} \tag{74}$$

¶ Let:
$p_j^{(k)}$ denote the p-values for the kth possibility in eq.(74).
$\mu_k(\alpha)$ denote the inner minimum evaluated with the kth option:

$$\mu_k(\alpha) = \sum_{j=1}^{3} (\widetilde{v}_{ij} - \alpha w_{ij{\rm s}}) p_j^{(k)} \tag{75}$$

We use a computer to evaluate the three options, looking for the minimal value. That is:

$$\mu(\alpha) = \min_k \mu_k(\alpha) \tag{76}$$

This produces the inverse of the robustness for the ith action.

3.5 Example

State	$\widetilde{p}_j$	$\widetilde{v}_{1j}$	$\widetilde{v}_{1js}$	$\widetilde{v}_{1j\ell}$	$\widetilde{v}_{2j}$	$\widetilde{v}_{2js}$	$\widetilde{v}_{2j\ell}$
No comp., $j=1$	0.6	10	7	11	9	7	11
Minor comp., $j=2$	0.15	8	5	11	7	5	8
Major comp., $j=3$	0.25	5	2	8	6	5	8

Table 2: Estimated probabilities and net values.

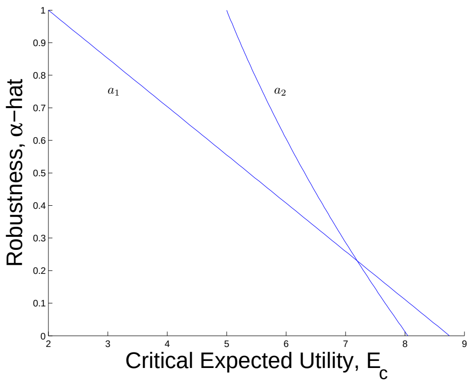


Figure 11: Robustness curves.

¶ We now consider a numerical example. The marketing data are shown in table 2.

¶ The best-estimates of the expected utilities of the two actions are:

$$\mathrm{E}(a_1, \widetilde{v}, \widetilde{p}) = 8.75 \tag{77}$$
$$\mathrm{E}(a_2, \widetilde{v}, \widetilde{p}) = 8.05 \tag{78}$$

- a_1 is preferred in terms of best-estimates of the expected utility.
- The robustnesses are zero at these net values:

$$\widehat{\alpha}(a_1, 8.75) = 0 \tag{79}$$
$$\widehat{\alpha}(a_2, 8.05) = 0 \tag{80}$$

- The robustness curves cross at:

$$(\mathrm{E}_\times, \alpha_\times) = (7.2, 0.22) \tag{81}$$

- a_2 is preferred at $\widehat{\alpha} > 0.22$ or at $E_c < 7.2$.

4 Value at Risk with Info-Gaps

¶ Source material: Yakov Ben-Haim, 2005, Value at risk with info-gap uncertainty, *Journal of Risk Finance,* vol. 6, #5, pp.388–403.

4.1 The Problem

¶ **The problem:**
- Fat tails: extreme adverse outcomes too frequent.
- Conditional volatility:
  Size of fluctuations varies in time.
- Predicting percentiles in financial markets:
  - 95th: Okay.
  - 99th: underestimated.

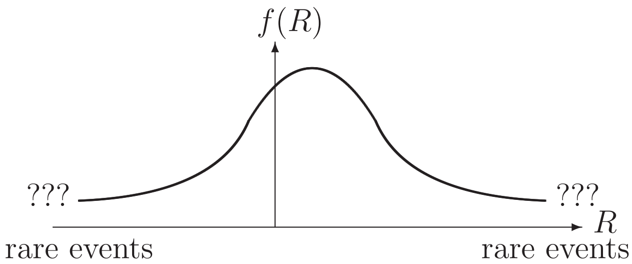


Figure 12: Uncertain tails of a distribution.

¶ **The opportunity:**
- Fat tails: extreme favorable outcomes occur too.
- Two sides to uncertainty:
  - Design **robustly** against failure: robust-satisficing.
  - Design **opportunely** for windfall: opportune-windfalling.

¶ **Two foci of uncertainty:**
- Statistical fluctuations:
  - Randomness, “noise”.
  - Estimation uncertainty.
- Info-gap uncertainty:
  - Surprises.
  - Structural changes.
  - Historical data used to predict future.

¶ We will use the concept from financial economics of **Value at Risk.**

¶ **Notation for VaR:**

W = initial value of portfolio, project, firm, etc.
V = revenue in one period.
$R = V/W$ = rate of return.
$f(R)$ = pdf for R.
- ∘ Highly uncertain.
- ∘ Large info-gaps.

$\widetilde{f}(R)$ = best estimate of $f(R)$.
- ∘ Statistical estimation.
- ∘ Historical data.

$R_\star$ = least acceptable rate of return (< 0).
- ∘ Reserve requirement.
- ∘ Insurance level.
- ∘ **Managerial decision.**

VaR = $R_\star W$ = max tolerable 1-period loss.

4.2 Uncertainty and Robustness

¶ **Info-gap uncertainty:**

$f(R)$ = unknown true pdf for R.
$\widetilde{f}(R)$ = best estimate of $f(R)$.
$\mathcal{F}(\alpha, \widetilde{f})$ = info-gap model for uncertainty in $f(R)$, fig. 13.
- Unbounded family of nested sets of pdfs.
- E.g. Unknown fractional error in $f(R)$:

$$\mathcal{F}(\alpha, \widetilde{f}) = \left\{ f(R) : \ f(R) \geq 0, \ \int_{-\infty}^{\infty} f(R) \, \mathrm{d}R = 1, \ \frac{\left| f(R) - \widetilde{f}(R) \right|}{\widetilde{f}(R)} \leq \alpha \right\}, \ \alpha \geq 0 \qquad (82)$$

- No worst case.
- Unbounded horizon of uncertainty, α.
- Info-gap uncertainty.

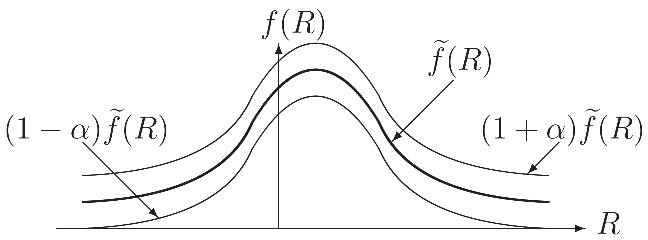


Figure 13: Uncertainty envelope at horizon of uncertainty α.

¶ **Failure probability:**

- Failure: loss more negative than **VaR**: $RW < R_\star W$.
- Given **VaR** $= R_\star W$, the failure probability is (fig. 14):

$$P_{\mathrm{f}}(f) = \int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R \tag{83}$$

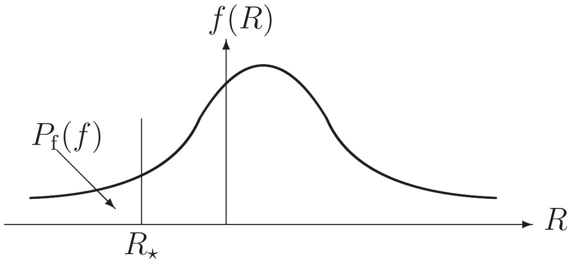


Figure 14: $q(c, f)$ is the cth quantile of $f(R)$.

- Requirement:

$$P_{\mathrm{f}}(f) \le c \tag{84}$$

  E.g., $c = 0.01$.
- **Problem:** $f(R)$ highly uncertain, especially on its tails.
- Estimated failure probability:

$$P_{\mathrm{f}}(\widetilde{f}) = \int_{-\infty}^{R_\star} \widetilde{f}(R)\,\mathrm{d}R \tag{85}$$

¶ **Robustness question:**

How wrong can $\widetilde{f}(R)$ be, without failure probability exceeding c?

¶ Info-gap robustness: Formulation.

- $q(c,f) = c$ th **quantile** of $f(R)$:

$$\int_{-\infty}^{q(c,f)} f(R)\,\mathrm{d}R = c \qquad (86)$$

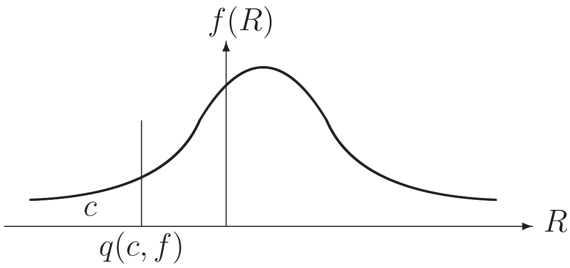


Figure 15: $q(c,f)$ is the cth quantile of $f(R)$.

- $R_\star$ = least acceptable rate of return (< 0):

$$R_\star = q(c, \widetilde{f}) \qquad (87)$$

- Estimated: $\mathbf{VaR}(c,\widetilde{f}) = q(c,\widetilde{f})W$
  If $\widetilde{f}$ is correct, then the prob of losing more than \$$\mathbf{VaR}(c,\widetilde{f})$ is less than c.
- True: $\mathbf{VaR}(c,f) = q(c,f)W$
- Robustness of $R_\star$ chosen with $\widetilde{f}(R)$:
  - Max tolerable info-gap in f.
  - Max α at which true **VaR** at confidence $1-c$, no worse than $R_\star W$:

$$\widehat{\alpha}(R_\star, c) = \max\left\{\alpha : \left(\min_{f\in\mathcal{F}(\alpha,\widetilde{f})} \mathbf{VaR}(c,f)\right) \geq R_\star W\right\} \qquad (88)$$

4.3 Deriving the Robustness Function

¶ We now derive the robustness function, defined in eq.(88) on p.25 as:

$$\widehat{\alpha}(R_\star, c) = \max\left\{\alpha : \left(\min_{f\in\mathcal{F}(\alpha,\widetilde{f})} \mathbf{VaR}(c,f)\right) \geq R_\star W\right\} \tag{89}$$

Recall that **VaR** is defined as:

$$\mathbf{VaR}(c,f) = q(c,f)W \tag{90}$$

Thus the robustness can be written:

$$\widehat{\alpha}(R_\star, c) = \max\left\{\alpha : \left(\min_{f\in\mathcal{F}(\alpha,\widetilde{f})} q(c,f)\right) \geq R_\star\right\} \tag{91}$$

where $q(c,f)$ is the cth quantile of $f(R)$.
The robustness is the max horizon of uncertainty so that the cth quantile of f exceeds $R_\star$.

¶ Recall the definition of the quantile:

$$\int_{-\infty}^{q(c,f)} f(R)\,\mathrm{d}R = c \tag{92}$$

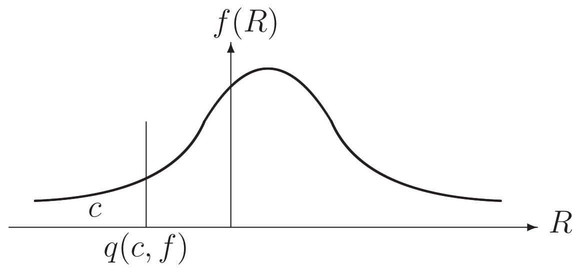


Figure 16: $q(c,f)$ is the cth quantile of $f(R)$.

¶ So, the min in eq.(91) occurs when the lower tail of $f(R)$ is as fat as possible:[1]

$$f(R) = (1+\alpha)\widetilde{f}(R) \tag{93}$$

[1] This is true for small c.

¶ The following two inequalities are equivalent, as illustrated in fig. 17:

$$q(c, f) \geq R_\star \quad \Longleftrightarrow \quad c \geq \int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R \tag{94}$$

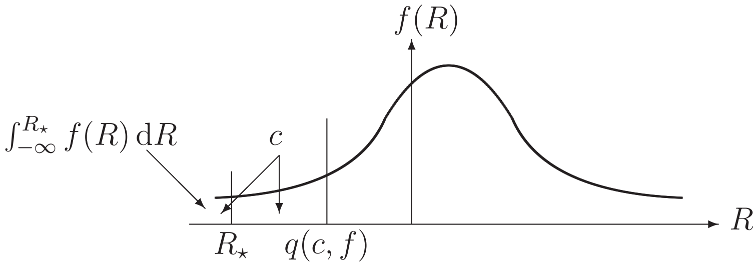


Figure 17: $q(c, f)$ is the cth quantile of $f(R)$.

¶ Using eqs.(93) and (94) the robustness is the max α at which:

$$c \geq \int_{-\infty}^{R_\star} (1+\alpha)\widetilde{f}(R)\,\mathrm{d}R \tag{95}$$

Solving this for α yields the robustness:

$$\widehat{\alpha}(R_\star, c) = \frac{c}{\displaystyle\int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R} - 1 \tag{96}$$

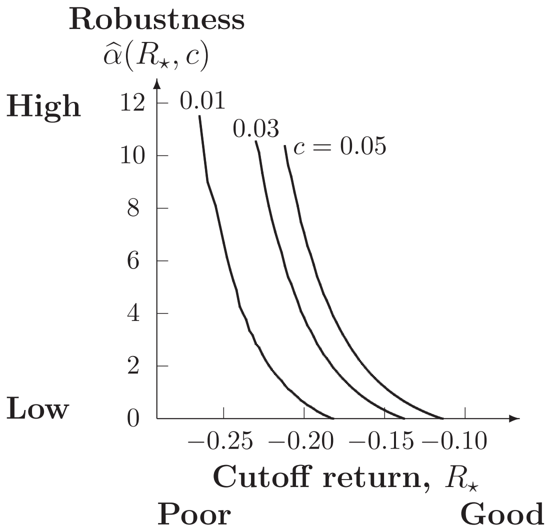
or zero if this is negative.

4.4 Numerical Example

¶ **Estimated pdf,** $\widetilde{f}(R)$, is normal: $\mu = 0.05$, $\sigma^2 = 0.01$.

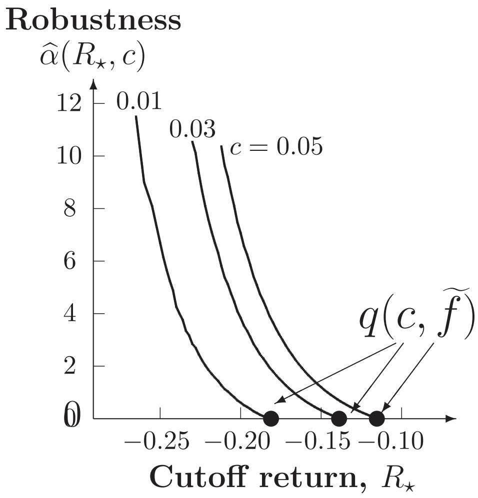
¶ **Trade-off:**
Robustness decreases as aspiration increases:

$$R_{\star 1} < R_{\star 2} < 0 \quad \text{implies} \quad \widehat{\alpha}(R_{\star 2}, c) \leq \widehat{\alpha}(R_{\star 1}, c) \tag{97}$$



¶ Best-model **VaR** is unreliable:

$$\widehat{\alpha}(R_\star^o, c) = 0 \quad \text{if} \quad R_\star^o = q(c, \widetilde{f}) \tag{98}$$



4.5 Opportuneness

¶ **Two faces of uncertainty:**
- Pernicious: threatening failure.
- Propitious: enabling windfall.

¶ We will now briefly study the propitious side of uncertainty.

¶ Recall:
- $q(c, f)$ is cth quantile of $f(R)$. Units: rate of return.
- $\mathbf{VaR}(c, f) = q(c, f)W$. Units: net return (or loss, if negative).

¶ **Robustness,** re-stated (eq.(88), p.25):

$$\widehat{\alpha}(R_\star, c) = \max\left\{\alpha:\ \left(\min_{f\in\mathcal{F}(\alpha,\widetilde{f})} \mathbf{VaR}(c, f)\right) \geq R_\star W\right\} \tag{99}$$

- $R_\star W$ is negative, a loss.
  We can think of it as a (negative) gain.
- $\widehat{\alpha}(R_\star, c)$ is the max horizon of uncertainty in the pdf, $f(R)$,
  up to which gain at least as large as $R_\star W$ is
  guaranteed with probability no less than $1 - c$.
- **Greatest horizon of uncertainty at which tolerable gain is guaranteed.**

¶ **Opportuneness:**
- **Lowest horizon of uncertainty at which wonderful gain is possible.**
- Windfall return: $R^\star W$, greater than $R_\star W$.

$$\widehat{\beta}(R^\star, c) = \min\left\{\alpha:\ \left(\max_{f\in\mathcal{F}(\alpha,\widetilde{f})} \mathbf{VaR}(c, f)\right) \geq R^\star W\right\} \tag{100}$$

- $\widehat{\beta}(R^\star, c)$ is the min horizon of uncertainty in the pdf, $f(R)$,
  up to which gain at least as large as $R_\star W$ is
  possible with probability no less than $1 - c$.

¶ **Immunity functions:**
- Robustness: immunity against failure. Bigger $\widehat{\alpha}$ is better.
- Opportuneness: immunity against windfall. Big $\widehat{\beta}$ is bad.

¶ Summary
- **VaR**: statistical assessment of risk.
- Foci of uncertainty:
  - Estimation error: randomness.
  - Info-gap uncertainty, surprise, past/future.
- Info-gap robustness:
  - Managing info-gaps.
  - Supplementing statistical **VaR**.
- Info-gap opportuneness:
  - Exploiting info-gaps.