Lecture Notes on

Military Budget Under Uncertainty

Yakov Ben-Haim
Yitzhak Moda'i Chair in
Technology and Economics
Faculty of Mechanical Engineering
Technion — Israel Institute of Technology
Haifa 32000 Israel
http://www.technion.ac.il/yakov
yakov@technion.ac.il

¶ Source material:

- Allocating Security Expenditures under Knightian Uncertainty: An Info-Gap Approach, Michael Ben-Gad, Yakov Ben-Haim and Dan Peled, Working Paper Series, Economics of National Security, Samuel Neaman Institute for Advanced Studies in Science and Technology, Technion.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

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1 Expected Utility

x = random variable with pdf $p(x)$.

q = decision.

$f(x,q)$ = outcome; e.g. utility.

$\mathrm{E}[f(x,q)]$ = average outcome; e.g. expected utility.

$q^\star$ = EU-maximizing decision, given knowledge of the pdf of x:

$$q^\star = \arg\max_q \mathrm{E}[f(x,q)] \tag{1}$$

Now consider uncertainty in the pdf of x.

$\widetilde{p}(x)$ = best estimate of $p(x)$.

$p(x)$ = unknown true pdf.

$\mathcal{U}(\alpha,\widetilde{p})$ = info-gap model for uncertainty in $\widetilde{p}(x)$.

$\alpha \geq 0$ is the unknown horizon of uncertainty.

E_c = lowest acceptable EU.

$\widehat{\alpha}(q,E_\mathrm{c})$ = robustness of decision q with required EU E_c:

$$\widehat{\alpha}(q,E_\mathrm{c}) = \max\left\{\alpha :\ \left(\max_{p\in\mathcal{U}(\alpha,\widetilde{p})} \mathrm{E}[f(x,q)|p]\right) \geq E_\mathrm{c}\right\} \tag{2}$$

Trade-off: robustness versus E_c.

Crossing of robustness curves.

2 Formulation

¶ Macro-economic model.

¶ **Expenditures:**

b_i = fraction of GDP spent on military expenditure of type i, $i = 1, \ldots, N$.

$b = (b_1, \ldots, b_N)^T$ = vector of military expenditures.

Total military expenditure:

$$0 \le \sum_{i=1}^{N} b_i \le 1 \tag{3}$$

b_{c} = fraction of GDP spent on non-military consumption. The available resources are:

$$b_{\mathrm{c}} = 1 - \sum_{i=1}^{N} b_i \tag{4}$$

¶ **Utility from consumption.**

Utility from non-military consumption is $u(b_{\mathrm{c}})$.

We assume positive marginal utility:

$$\frac{\mathrm{d}u(b_{\mathrm{c}})}{\mathrm{d}b_{\mathrm{c}}} > 0 \tag{5}$$

Available utility from consumption is $u_{\mathrm{c}} = u(b_{\mathrm{c}})$.

¶ **War-related damage.**

D = fractional loss in resources resulting from belligerent action, where $0 \le D \le 1$.

$p(D|b)$ = pdf of damage D conditioned on military budget b.

$p(D|b)$ is highly uncertain.

$\widetilde{p}(D|b)$ is best known estimate of $p(D|b)$.

¶ **Probability of war.**

P_{w} = probability of war.

P_{w} is highly uncertain.

$\widetilde{P}_{\mathrm{w}}$ is best known estimate of P_{w}.

¶ **Info-gap models of uncertainty.**

- If both $p(D|b)$ and P_{w} are uncertain, then a fractional-error info-gap model is:

$$\begin{aligned}\mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\mathrm{w}}) = \Big\{ p(D|b),\, P_{\mathrm{w}} : \quad & p(D|b) \in \mathcal{P},\ |p(D|b) - \widetilde{p}(D|b)| \le \alpha \widetilde{p}(D|b),\ \text{for all } D \\ & 0 \le P_{\mathrm{w}} \le 1,\ |P_{\mathrm{w}} - \widetilde{P}_{\mathrm{w}}| \le \alpha \widetilde{P}_{\mathrm{w}} \Big\}, \quad \alpha \ge 0 \end{aligned} \tag{6}$$

$\mathcal{P}$ = all legitimate pdfs.

- If only $p(D|b)$ is uncertain then:

$$\mathcal{U}(\alpha, \widetilde{p}) = \left\{ p(D|b) :\ p(D|b) \in \mathcal{P},\ |p(D|b) - \widetilde{p}(D|b)| \le \alpha \widetilde{p}(D|b),\ \text{for all } D \right\}, \quad \alpha \ge 0 \tag{7}$$

¶ **Expected utility** of expenditures b:

- If war occurs, then the expected utility from uncertain damage pdf $p(D)$ is:

$$\int_0^1 u\left[(1-D)b_{\mathrm{c}}\right] p(D|b)\, \mathrm{d}D \tag{8}$$

- If war does not occur, then the expected utility is:

$$u_{\mathrm{c}} = u(b_{\mathrm{c}}) \tag{9}$$

- The probability of war is:

$$P_{\mathrm{w}} \tag{10}$$

- The probability of no war is:

$$1 - P_{\mathrm{w}} \tag{11}$$

- Combining eqs.(8)–(11):

$R(b|p, P_{\mathrm{w}})$ = expected utility from expenditures b if the conditional pdf of the damage D is $p(D|b)$ and the probability of war is P_{w}:

$$R(b|p, P_{\mathrm{w}}) \;=\; \left(\int_0^1 u\left[(1-D)b_{\mathrm{c}}\right] p(D|b)\, \mathrm{d}D \right) P_{\mathrm{w}} + (1 - P_{\mathrm{w}}) u_{\mathrm{c}} \tag{12}$$

- R_{c} is the lowest acceptable reward. It is a reward aspiration or a 'reservation' reward.

Satisfice reward. Require:

$$R(b|p, P_{\mathrm{w}}) \ge R_{\mathrm{c}} \tag{13}$$

¶ **Robustness.**

$\widehat{\alpha}(b, R_{\rm c})$ is the robustness of expenditures b with reward-aspiration $R_{\rm c}$:

$$\widehat{\alpha}(b, R_{\rm c}) = \max \left\{ \alpha : \ \left(\min_{p, P_{\rm W} \in \mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})} R(b|p, P_{\rm w}) \right) \geq R_{\rm c} \right\} \tag{14}$$

¶ **Trade-off of robustness against utility,** fig. 1:

$$R_{\rm c} > R'_{\rm c} \quad \Longrightarrow \quad \widehat{\alpha}(b, R_{\rm c}) \leq \widehat{\alpha}(b, R'_{\rm c}) \tag{15}$$

¶ **Zero utility of nominal reward,** fig. 1:

$$R_{\rm nom} = R(b|\widetilde{p}, \widetilde{P}_{\rm w}) \quad \Longrightarrow \quad \widehat{\alpha}(b, R_{\rm nom}) = 0 \tag{16}$$

Utility R_2 less desirable but more reliable than $R_{\rm nom}$.

Utility R_1 less desirable but more reliable than R_2.

¶ **Preferences:**

$$b \succ b' \quad \text{if} \quad \widehat{\alpha}(b, R_{\rm c}) \geq \widehat{\alpha}(b', R_{\rm c}) \tag{17}$$

¶ **Reversal of preferences,** fig. 2.

$$b \succ b' \quad \text{at} \quad R_1 \qquad\qquad\qquad b' \succ b \quad \text{at} \quad R_2 \tag{18}$$

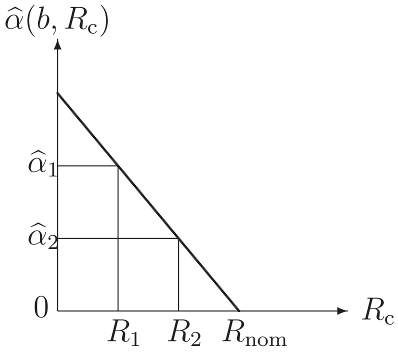

Figure 1: Robustness curve for military budget b.

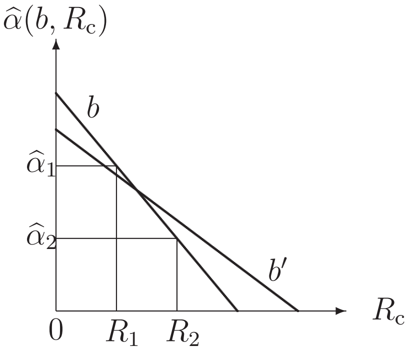

Figure 2: Robustness curves for military budgets b and b'.

¶ Utility maximization.

- Given best estimates $\widetilde{p}$ and $\widetilde{P}_{\rm w}$, the utility-maximizing budget is:

$$b^\star = \arg\max_b R(b|\widetilde{p}, \widetilde{P}_{\rm w}) \qquad (19)$$

- Zero robustness of maximal utility, eq.(16), p.5:

$$R_{\rm c} = R(b^\star|\widetilde{p}, \widetilde{P}_{\rm w}) \quad \Longrightarrow \quad \widehat{\alpha}(b, R_{\rm c}) = 0 \qquad (20)$$

¶ Robust-satisficing:

$$\widehat{b}(R_{\rm c}) = \arg\max_b \widehat{\alpha}(b, R_{\rm c}) \qquad (21)$$

¶ Robustness premium:

$$\Delta\widehat{\alpha} = \widehat{\alpha}[\widehat{b}(R_{\rm c}),\ R_{\rm c}] - \widehat{\alpha}(b^\star, R_{\rm c}) \qquad (22)$$

Necessarily:

$$\Delta\widehat{\alpha} \ge 0 \qquad (23)$$

Usually strictly positive.

¶ Utility premium:

$$\Delta R(p, P_{\rm w}) = R[\widehat{b}(R_{\rm c})|p, P_{\rm w}] - R[b^\star|p, P_{\rm w}] \qquad (24)$$

Necessarily:

$$\Delta R(\widetilde{p}, \widetilde{P}_{\rm w}) \le 0 \qquad (25)$$

$\Delta R(p, P_{\rm w})$ can be strictly positive for other realizations of p and $P_{\rm w}$.

3 Example: Two Types of Military Expenditure

3.1 Formulation

¶ Two military expenditures:

b_1 = expenditure on intelligence.

b_2 = expenditure on armor.

¶ Stylized facts on b and probability distribution of war damage D:

- Mean D generally declines as the $b_1 + b_2$ increases:

$$\frac{\partial \mathrm{E}(D)}{\partial (b_1 + b_2)} < 0 \tag{26}$$

- Mean damage and extreme damage are antagonistic at fixed total military expenditure:

$$\left.\frac{\partial \mathrm{E}(D)}{\partial b_1}\right|_{b_1+b_2} \left.\frac{\partial \mathrm{Prob}(D > 0.4)}{\partial b_1}\right|_{b_1+b_2} < 0 \tag{27}$$

¶ Best estimate of damage distribution:

$$\widetilde{p}(D|b) = \kappa \varepsilon \left(\frac{b_1}{b_2}\right) \exp[-\theta(b_1 + b_2) \cdot D] + \kappa \left[1 - \varepsilon\left(\frac{b_1}{b_2}\right)\right] \frac{D^{a-1}(1-D)^{b-1}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \tag{28}$$

where κ is a normalization constant, $\theta = 25$ and:

$$\varepsilon\left(\frac{b_1}{b_2}\right) = 0.6 - 0.05\frac{b_1}{b_2} \tag{29}$$

$$a = 10\frac{b_1^2 + b_2^2}{b_1} \tag{30}$$

$$b = 10\frac{b_1 + b_2}{b_1} \tag{31}$$

3.2 Robustness Function

¶ **Expected utility** if war occurs, based on the estimated pdf of damage, is:

$$\widetilde{r} = \int_0^1 u\,[(1-D)b_{\mathrm{c}}]\,\widetilde{p}(D|b)\,\mathrm{d}D \tag{32}$$

Utility if war does not occur is $u_{\mathrm{c}} = u(b_{\mathrm{c}})$.

We assume:

$$\widetilde{r} < u_{\mathrm{c}} \tag{33}$$

Total estimated expected utility, based on best estimates:

$$\widetilde{R} = P_{\mathrm{w}}\widetilde{r} + (1-P_{\mathrm{w}})u_{\mathrm{c}} \tag{34}$$

¶ **Robustness** for $\widehat{\alpha} \le 1$ and based on the info-gap model of eq.(6) and on assumption (33), is derived in the appendix, section 4:

$$\widehat{\alpha}(b, R_{\mathrm{c}}) = \begin{cases} \dfrac{(\widetilde{r} - u_{\mathrm{c}} - \delta_r)\widetilde{P}_{\mathrm{w}} + \sqrt{(\widetilde{r} - u_{\mathrm{c}} - \delta_r)^2\widetilde{P}_{\mathrm{w}}^2 + 4\delta_r\widetilde{P}_{\mathrm{w}}(\widetilde{R} - R_{\mathrm{c}})}}{2\delta_r\widetilde{P}_{\mathrm{w}}} & \text{if } \widetilde{R} \ge R_{\mathrm{c}} \\ 0 & \text{else} \end{cases} \tag{35}$$

where:

$$\widetilde{r}_1 = \int_0^{D_{\mathrm{m}}} u\,[(1-D)b_{\mathrm{c}}]\,\widetilde{p}(D|b)\,\mathrm{d}D \tag{36}$$

$$\widetilde{r}_2 = \int_{D_{\mathrm{m}}}^1 u\,[(1-D)b_{\mathrm{c}}]\,\widetilde{p}(D|b)\,\mathrm{d}D \tag{37}$$

$$\delta_r = \widetilde{r}_1 - \widetilde{r}_2 \tag{38}$$

$$D_{\mathrm{m}} = \text{median of } \widetilde{p}(D|b) \tag{39}$$

Positive marginal utility implies $\widetilde{r}_1 > \widetilde{r}_2$.

Hence $\delta_r > 0$.

Hence, with eq.(33): $\widetilde{r} - u_{\mathrm{c}} - \delta_r < 0$.

¶ **Robustness** based on the info-gap model of eq.(7), without requiring assumption (33), is:

$$\widehat{\alpha}(b, R_{\mathrm{c}}) = \begin{cases} \dfrac{(\widetilde{r} - u_{\mathrm{c}})\widetilde{P}_{\mathrm{w}} + u_{\mathrm{c}} - R_{\mathrm{c}}}{\delta_r\widetilde{P}_{\mathrm{w}}} & \text{if } (\widetilde{r} - u_{\mathrm{c}})\widetilde{P}_{\mathrm{w}} + u_{\mathrm{c}} \ge R_{\mathrm{c}} \\ 0 & \text{else} \end{cases} \tag{40}$$

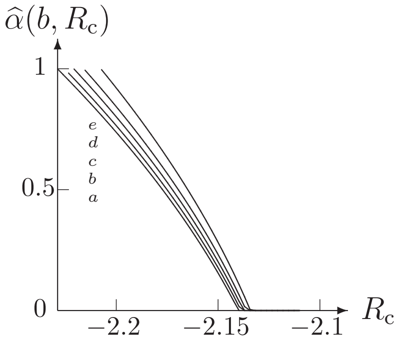

Figure 3: Robustness curves for $b_1 + b_2 = 0.1$, $b_1 = 0.03$ (*a*), 0.04 (*b*), 0.05 (*c*), 0.06 (*d*), 0.07 (*e*).

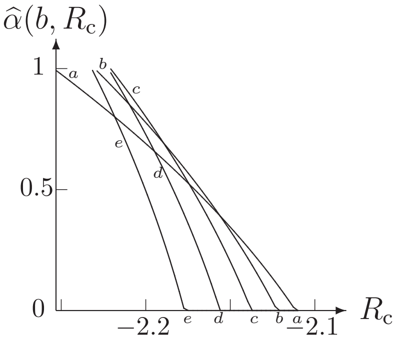

Figure 4: Robustness curves for $b_1 = b_2 = 0.03$ (*a*), 0.04 (*b*), 0.05 (*c*), 0.06 (*d*), 0.07 (*e*).

3.3 Numerical Results

¶ Formulation.

- We use the constant-risk-aversion utility function:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{41}$$

  where $\gamma > 0$ and, in the following calculations, $\gamma = 1.5$.

- The nominal pdf is defined in eqs.(28)–(31).
- We explore a range of values of military budget, $b = (b_1, b_2)$.
- Robustness curves from eq.(35) are displayed in figs. 3 and 4.
- The median of $\widetilde{p}(D)$ ranges from about 0.04 to about 0.06.

  I.e., the nominal pdf entails very low probability for high damage if war occurs.

- The estimated probability of war is $\widetilde{P}_w = 0.1$.

¶ **Fixed total military budget, fig. 3.**

- Robustness $\widehat{\alpha}(b, R_{\rm c})$ vs. demanded expected utility $R_{\rm c}$.
- $b_1 + b_2 = 0.1$. b_1 = intelligence. b_2 = armor.
- Allocations for intelligence (b_1) and armor (b_2) differ:

  From lower to upper curve, a to e, b_1 varies from 0.03 to 0.07 in steps of 0.01.
- Trade-off between aspiration for utility, $R_{\rm c}$, and robustness against uncertainty $\widehat{\alpha}(b, R_{\rm c})$:

  Higher aspirations are less reliable than lower aspirations, eq.(15), p.5.
- Robustness vanishes at sufficiently large aspiration, specifically:

  at the expected utility based on the best estimates of the probabilities, as in eq.(20), p.6.
- Robustness increases as armor allocation decreases, at fixed total budget.

  Thus intelligence should be favored over armour, for the values examined.

¶ **Variable military budget, fig. 4.**

- Robustness $\widehat{\alpha}(b, R_{\rm c})$ vs. demanded expected utility $R_{\rm c}$.
- $b_1 = b_2 = 0.03$ to 0.07 in steps of 0.01 for curves a to e.
- Consider intersection of $\widehat{\alpha}(b, R_{\rm c})$ with the $R_{\rm c}$-axis:
  - Occurs when $R_{\rm c}$ equals expected utility of the best-estimated probabilities.
  - Expected utility increases as the total military budget decreases (from curve e to a).
  - From max expected utility we would choose the least total military budget, curve a.
  - But the robustness is zero on the $R_{\rm c}$-axis, as expected from eq.(20).
  - "Migrate" up the robustness curves to reach reliable values of utility.
- Robustness curves cross:
  - Curve b crosses curve a at very low robustness.
  - Curve c crosses curve a at slightly greater robustness.
  - If $\widehat{\alpha} = 0.6$ has adequate utility on curve a, then
    - — the same utility has greater robustness on curves b and c, or
    - — curves b and c provide greater utility than curve a at robustness $\widehat{\alpha} = 0.6$.
  - Crossing of robustness curves may entail reversal of preferences.

4 Appendix: Derivation of the Robustness Function

In this appendix we derive the robustness function in eq.(35) based on assumption (33) and for values of the robustness not in excess of unity: $\widehat{\alpha} \le 1$. We make no assumptions about the utility function $u(c)$ other than that the marginal utility is positive: $\dot{u}(c) > 0$.

The robustness is defined in eq.(14). The main task is to find the pdf of the damage, $p(D|b)$, which, at horizon of uncertainty α, minimizes the expected utility $R(b|p, P_{\rm w})$ defined in eq.(12). Because the marginal utility is positive it is evident that $R(b|p, P_{\rm w})$ is minimized by that pdf in $\mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})$ which puts as much weight as possible at large damage and as little weight as possible at low damage. For the fractional-error info-gap model in eq.(6) one readily shows that $\min\limits_{\alpha} R(b|p, P_{\rm w})$ occurs with the following pdf:

$$p(D|b) = \begin{cases} (1-\alpha)\widetilde{p}(D|b) & \text{if } D \le D_{\rm m} \\ (1+\alpha)\widetilde{p}(D|b) & \text{else} \end{cases} \tag{42}$$

where $D_{\rm m}$ is the median of the estimated pdf $\widetilde{p}(D|b)$ and where $\alpha \le 1$.

If $\alpha > 1$ then $\min\limits_{\alpha} R(b|p, P_{\rm w})$ occurs with the following pdf:

$$p(D|b) = \begin{cases} 0 & \text{if } D \le D_{\rm s} \\ (1+\alpha)\widetilde{p}(D|b) & \text{else} \end{cases} \tag{43}$$

where $D_{\rm s}$ satisfies:

$$(1+\alpha)\int_{D_{\rm s}}^{1} \widetilde{p}(D|b)\,{\rm d}D = 1 \tag{44}$$

In other words, $D_{\rm s}$ is the $1 - 1/(1+\alpha)$ quantile of $\widetilde{p}(D|b)$.

We will consider only the case $\alpha \le 1$. The derivation of robustness in excess of unity is analogous.

The utility $R(b|pP_{\rm w})$ in eq.(12), evaluated with the pdf in eq.(42), is:

$$R(b|pP_{\rm w}) = [\widetilde{r} - \delta_r\alpha - u_{\rm c}]\, P_{\rm w} + u_{\rm c} \tag{45}$$

where $\widetilde{r}$ and δ_r are defined in eqs.(32) and (38) and $u_{\rm c}$ is the utility if war does not occur, $u(b_{\rm c})$. The term $\widetilde{r} - \delta_r\alpha - u_{\rm c}$ is negative so the minimizing value of $P_{\rm w}$ in $\mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})$ is $(1+\alpha)\widetilde{P}_{\rm w}$. Thus the minimum expected utility, up to horizon of uncertainty α, is:

$$\min_{p, P_{\rm W} \in \mathcal{U}(\alpha, \widetilde{p}, \widetilde{P}_{\rm w})} R(b|p, P_{\rm w}) = [\widetilde{r} - \delta_r\alpha - u_{\rm c}]\,(1+\alpha)\widetilde{P}_{\rm w} + u_{\rm c} \tag{46}$$

Denote this minimum $\mu(\alpha)$, which decreases monotonically as α increases because $\delta_r > 0$. The robustness is, according to the definition in eq.(14), the greatest value of α up to which $\mu(\alpha)$ is no less than R_{c}. That is, the robustness is the lowest non-negative solution for $\widehat{\alpha}$ of:

$$\mu(\widehat{\alpha}) = R_{\mathrm{c}} \tag{47}$$

This is a quadratic equation in α whose least non-negative root is eq.(35).

The derivation of the robustness function in eq.(40) is obtained by equating eq.(45) to R_{c} and solving for α, with $\widetilde{P}_{\mathrm{w}}$ instead of P_{w}.